Alexandria Engineering Journal (2022) 61, 10197–10218

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# Stokes' second problem and oscillatory Couette flow for a two-layer fluid: Analytical solutions

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Received 27 November 2021; revised 26 February 2022; accepted 10 March 2022
Available online 1 April 2022

**KEYWORDS**

Stokes' second problem;
Oscillatory Couette flow;
Two-layer fluid;
The Laplace transform method;
Transient velocity;
Steady-state velocity

**Abstract** The unsteady motion of a two-layer fluid induced by oscillatory motion of a flat plate along its length is mathematically analyzed. Two cases are considered: (i) the two-layer fluid is bounded only by the oscillating plate (Stokes' second problem), and (ii) the two-layer fluid is confined between two parallel plates, one of which oscillates while the other is held stationary (oscillatory Couette flow). In each of the Stokes and Couette cases, both cosine and sine oscillations of the plate are considered. It is assumed that the fluids are immiscible, and that the flat interface between the fluids remains flat for all times. Solutions to the initial-boundary value problems are obtained using the Laplace transform method.The results obtained for the velocity fields for the flows are new and complete. Steady periodic and transient velocity fields are explicitly presented. Transient and steady-state shear stresses at the boundaries of the flows are calculated; the existing literature lacks any such results. The results derived in this paper retrieve previously known results for corresponding single-layer flows. Furthermore, the results obtained are illustrated taking particular example of each of the Stokes' problem and the Couette flow. With the help of the illustrations, some physical insights into the flows of the particular problems are gained. Again, the results obtained could also be applicable to a problem of heat conduction in a composite solid with sinusoidal temperature variation on the surface.



## 1. Introduction

In fluid mechanics, Stokes' second problem refers to the motion of a semi-infinite incompressible viscous fluid induced by an oscillating flat plate [1,2]. However, Zeng and Weinbaum[3] has called it as Stokes' first problem. In Stokes' problem, the fluid is bounded only by the oscillating plate. Again, when a fluid is bounded by two parallel plates, one of which oscillates in its own plane while the other is held stationary, the problem is termed oscillatory Couette flow [4]. The study of Stokes' second problem finds its applications in fields such as chemical engineering, medical and biomedical sciences, biomechanics, micro- and nano-technology, geophysical flows, and heat conduction problems [5–8]. It is worth mentioning here that Stokes' second problem has its counterparts in prob-

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☆ Peer review under responsibility of Faculty of Engineering, Alexandria University.

https://doi.org/10.1016/j.aej.2022.03.023
1110-0168

**Nomenclature**

| | |
|---|---|
| $\tilde{u_1}$ | velocity of the lower fluid in s-domain |
| $\tilde{u_2}$ | velocity of the upper fluid in s-domain |
| $H$ | distance between the plates |
| $h$ | thickness of the lower fluid |
| $s$ | Laplace transform variable |
| $t$ | time |
| $U_0$ | the plate velocity amplitude |
| $u_1$ | velocity of the lower fluid |
| $u_2$ | velocity of the upper fluid |
| $u_{1s}$ | steady-state velocity of the lower fluid |
| $u_{1t}$ | transient velocity of the lower fluid |
| $u_{2s}$ | steady-state velocity of the upper fluid |
| $u_{2t}$ | transient velocity of the upper fluid |
| *Greek letters* | |
| $\mathscr{L}$ | Laplace transform operator |
| $\mathscr{L}^{-1}$ | inverse Laplace transform operator |
| $\mu$ | dynamic viscosity |
| $\mu_2$ | dynamic viscosity of the upper fluid |
| $\mu_1$ | dynamic viscosity of the lower fluid |
| $v$ | kinematic viscosity |
| $v_1$ | kinematic viscosity of the lower fluid |
| $v_2$ | kinematic viscosity of the upper fluid |
| $\omega$ | frequency of oscillations of the plate |
| $\tau$ | shear stress |
| *Subscripts* | |
| 1 | lower fluid |
| 2 | upper fluid |
| $s$ | steady-state |
| $t$ | transient |
| $w$ | wall |

lems: acoustic streaming past an oscillating body, and settled boundary layer with fluctuating incident fluid velocity [9]. Non-steady flow of a viscous fluid adjacent to an oscillatory plate has drawn attention of many researchers. Erdogan [10] and Fetecau et al. [11] have dealt with Stokes' second problem in-depth. They have presented complete solution for the velocity field, obtained by the Laplace transform method. It is to be noted here that the complete solution contains transient and steady-state solutions. The original solution of the now-classical Stokes' second problem contains only steady-state solution, not valid for small values of time $t$. Again, Khaled and Vafai[4] have studied Stokes' second problem and oscillatory Couette flow with slip boundary condition on the plate(or plates). Relevantly, a recent study by Sherwood [12] is worth mentioning here. He has examined unsteady flow of a viscous fluid caused by an oscillating porous wall. All the works mentioned above are for Newtonian fluid. For non-Newtonian fluids, in the present context, the works of Rajagopal [13], Ai and Vafai [5], and Asghar *et al.* [14] are worth mentioning, among others. Also, a more recent study concerning viscoelastic fluids is worth noting here. Ortin [15] has investigated motions of viscoelastic fluids induced by an oscillating wall. He has carried out the study from the perspective of Stokes layers for the fluids. He has considered flows in both semi-infinite and wall-bounded domains. Again, some researchers have considered the convective flow of a non-Newtonian fluid over an oscillating plate in a rotating system. Mahanthesh *et.al.* [16] have examined nonlinear convective flow of Casson fluid over an oscillating plate subject to non-coaxial rotation of the plate and the fluid at infinity. They have adopted the Laplace transfom method to obtain an analytical solution to the problem. A similar work for hybrid nanofluids has been reported by Ashlin and Mahanthesh [17].

Now we turn to works concerning flows in a wall-bounded domain. Note that we have already mentioned the study by Khaled and Vafai[4] dealing with oscillatory Couette flow with slip boundary condition at the plates. Oscillatory Couette flow subjected to applied magnetic field or/and rotation has received considerable attention of researchers due to geophysical and engineering applications of the studies. Vajravelu [18] has studied the effect of applied magnetic field on the flow of an electrically conducting fluid between two parallel plates, one of which oscillates in its own plane while the other is stationary. Again, Mazumder [19] and Ganapathy [20] have investigated oscillatory Couette flow in a rotating system. The effects of both applied magnetic field and rotation on oscillatory Couette flow have been examined by Seth *et.al.* [21] and Singh [22]. It is worth noting that the works studying the effects of magnetic field or/and rotation have reported only the steady-state results. In another paper, Seth and Singh [23] have studied hydromagnetic flow between parallel plates, one of which oscillates in its own plane while the other is stationary, in a rotating system. These authors have employed the Laplace transform method to tackle the mathematical problem, obtaining both transient and steady-state results. Their work has been extended by Seth *et.al.* [24] where the induced magnetic field has been taken into account. The works dealing with motions in a wall-bounded domain that have been mentioned so far concern Newtonian fluid. On the other hand, for non-Newtonian fluids, oscillatory Couette flow subjected to applied magnetic field and rotation has been studied by Hayat *et.al.* [25] and Hayat *et.al.* [26], among others. In the light of the above discussion, we can say that a substantial amount of research has been applied to explore the flow of a single-layer fluid induced by oscillatory motion of a flat plate in its own plane. However, as far as we are aware, unsteady flow of a two-layer fluid caused by time-dependent movement of a boundary has received much less attention, noting that such problems may arise in some practical situations. Recently, Ng [27] has investigated change of Navier slip length with respect to time in starting flows using a two-layer flow model between parallel plates, where the flow is caused by impulsive motion of one of the plates along its length. In fact, the flow model is an extension of classical Stokes' first problem [2] for a single-layer fluid to the case of unsteady Couette flow of a two-layer fluid due to sudden motion of one of the plates.

In order to obtain the velocity fields for the two layers of fluid, he has utilized the result given in [28] concerning heat conduction in a composite solid. Pertinently, the work by Wang [29] dealing with pressure-driven oscillatory flow of a two-layer fluid is worth mentioning here.

The object of the present paper is to theoretically study the flow of a two-layer fluid induced by oscillatory motion of a flat plate in its own plane. Note that a two-layer fluid occurs in chemical engineering, lubricated piping, lithographic printing, oil industry, microfluidics, and nuclear reactor [30–41]. Note also that interaction of a viscous fluid with oscillatory shearing motion of a wall is found in many engineering applications[4]. In this study, we consider that the fluids that form the layers are Newtonian. We consider flows in both semi-infinite (Stokes' problem) and wall-bounded (Couette flow) domains. To the best of authors' knowledge, the existing literature lacks any complete velocity fields for the flows and any results for shear stresses at the boundaries of the flows. The results obtained in the present study for the velocity fields and wall shear stresses are complete and will be accounted in the literature for the first time. Note that a complete velocity field or shear stress consists of transient and steady-state parts. It should be mentioned here that steady-state velocity fields for the lower and upper fluids for the Couette flow case are given in Coward and Papageorgiou[42] and Halpern and Frenkel [43]. In both the works, the separation of variables method was used to obtain solution for the velocity fields. The aim of their works was to study the stability of oscillatory two-phase Couette flow. It is important to note that a steady-state velocity field is not a complete velocity field when a fluid starts moving from rest. A complete velocity field contains both transient and steady-state velocity fields. A result representing only steady-state velocity field is not valid for small values of time $t$. Moreover, the results in [42,43] are subject to restrictions respectively on densities and viscosities of the fluids of the layers. In the present study, for both the Stokes' problem and the Couette flow cases, we employ the Laplace transform method to obtain results for velocity fields, giving results for both transient and steady-state velocity fields. Note that the steady-state velocity fields (for the lower and upper fluids) obtained here for the Couette flow case are not similar to those reported in [42,43]. It is worth pointing out that the results obtained in the present study retrieve, as special cases, related results for a single-layer fluid motion reported in [4]. We believe the current study will deepen our understanding of the flow of a two-layer fluid caused by oscillatory motion of a wall in an engineering application. Besides, the results obtained here are also applicable to problems of heat conduction in two-layer composite solids with sinusoidal temperature variation on their surfaces. Further, the present study may provide a basis for some important future researches.

The present investigation may be considered as the starting point of a future study dealing with the motion of a two-layer fluid between parallel plates, one of which is oscillating in its own plane and the other is stationary, with slip boundary condition at the plates. We believe such a study could be applicable to microfluidics. Note that a two-layer fluid may occur in microfluidics [37–39], and many microfluidics have oscillating parts [44,45]. Again, the present work may provide a basis for future researches on Stokes' second problem and oscillatory Couette flow for a two-layer fluid where the fluids that form the layers are non-Newtonian fluids or viscoelastic fluids. It is our believe that such studies could be helpful in rheological studies of some fluids. Note that oscillatory shear is one of the methods that are applied to study rheological property of fluids [46–48], and some fluids behave as two-layer fluids during rheological studies using oscillatory shear [49]. Again, as we have hinted above, the current study is also applicable to a problem of heat conduction in a composite solid with the conditions as follows. The composite solid is initially at a uniform zero temperature and then suddenly, the surface of the solid comes into contact with a heat source with sinusoidal temperature variation. Note that there is an analogy between viscous diffusion in liquids and unsteady heat conduction in solids. Note also that a composite solid is formed by attaching together slaps of two different materials. Relevantly, Carslaw and Jaeger [28] have studied heat conduction in semi-infinite and finite composite solids where in each of the cases the surface of the solid suddenly comes into contact with a heat source with constant temperature of certain amount. They have also examined heat-conduction in a single-layer solid where the surface of the solid suddenly comes into contact with a heat source with sinusoidal temperature variation. It is worth mentioning here that Parasnis [50] has investigated steady-state heat conduction in a semi-infinite composite solid where the surface temperature varies sinusoidally with time $t$. But, as far as we are aware, the literature lacks any study of heat conduction in a finite composite solid with sinusoidal temperature variation on the surface. Further, the literature lacks any exhaustive work dealing with heat conduction in a semi-infinite composite solid with sinusoidal temperature variation on the surface.

In this work, we derive analytical solutions for two cases of unsteady motion of a two-layer fluid induced by sinusoidal oscillation of a flat plate in its own plane. We consider two cases: (i) the two-layer fluid is bounded only by the oscillating plate (Stokes' second problem) (see sketch in Fig. 1), and (ii) the two-layer fluid is confined between two parallel plates, one of which oscillates while the other is held stationary (oscillatory Couette flow) (see sketch in Fig. 2). In each of the cases, we consider both cosine and sine oscillations of the plate. The fluids of the two layers are Newtonian and have different viscosities, densities, and thickness. We assume that the fluids are immiscible, and that the flat interface between the fluids remains flat for all times. Here we note what follows. The current study could be applicable to a case where the interface is flat or the deviation of the interface from flat shape is small. Note also that the analytical results presented in this paper may be used for validation of future numerical works dealing with the following problems: Stokes' second problem and oscillatory Couette flow for a two-layer fluid with wavy interface between the layers of fluid. We employ the Laplace transform method to solve the initial-boundary value problems related to the two cases mentioned here.We use the method to solve the mathematical problems because it gives complete, analytical solutions valid for both small and large values of time $t$. It is important to note here that when a fluid starts moving from rest the velocity field contains a transient alongside the steady-state part.The transient disappears gradually (or rapidly) as time progresses. For both the Stokes and Couette cases, we present analytical results for velocity fields for starting and steady periodic flows. The result for a starting flow is the sum of transient solution and steady-state solution. The result for a starting flow is valid for small values of time $t$.

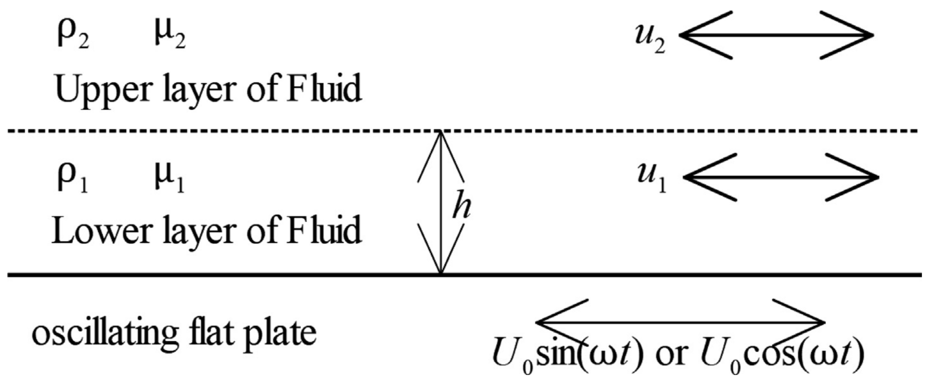



**Fig. 1** Schematic diagram for Stokes' second problem for a two-layer fluid. The solid line represents the plate, and the broken line is the interface between the fluids.

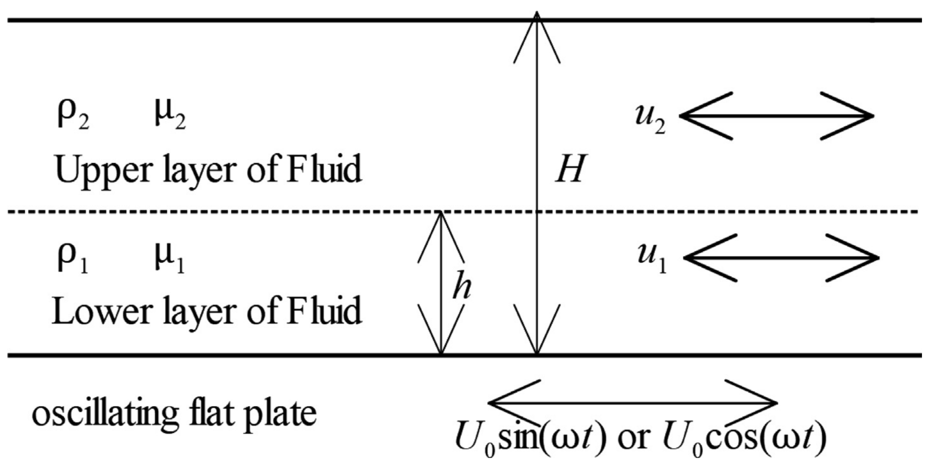



**Fig. 2** Schematic diagram for oscillatory Couette flow for a two-layer fluid. The solid lines represent the plates, and the broken line is the interface between the fluids.

Whereas the steady-state solution represents the time periodic motion of the fluid and is valid for large values of time $t$. We calculate transient and steady-state shear-stresses at the boundaries of the flows. We recover related previously known results for single-layer flows from the results obtained in this study. Also, as special cases, we deduce some results for oscillatory Couette flow for a single-layer fluid from the results obtained here. To the best of authors' knowledge, these results were not previously reported in the literature. We illustrate the results obtained in this study by taking particular example of each of the Stokes' problem and the Couette flow. The illustrations help us gain some physical insights into the flows of the problems taken as particular examples.

The remaining part of the paper is organized into four sections. Section 2 deals with the Stokes' second problem case, while Section 3 concerns the case of oscillatory Couette flow. Section 4 presents results and illustrative examples. And Section 5 concludes the paper.

## 2. Stokes' second problem for a two-layer fluid

### 2.1. Mathematical Formulation

Consider two superposed layers of two incompressible, viscous fluids over a flat plate that coincides with the $x$-$z$ plane of the Cartesian co-ordinate system $(x, y, z)$. The $y$-axis is the coordinate normal to the plate. Both of the two fluids that form the layers are Newtonian. The fluids are of different viscosities and densities. The two fluids of the layers are immiscible. Suppose that the lower fluid occupies the region $0 \leqslant y \leqslant h, h$ being a positive real number. And the upper fluid fills the region $h \leqslant y < \infty$. We consider that the fluids and the plate are initially at rest and then the plate starts to oscillate in its own plane along $x$-axis with velocity $U_0 \cos(\omega t)$ or $U_0 \sin(\omega t)$, where $U_0, \omega$, and $t$ being the plate velocity amplitude, frequency of oscillations, and the time, respectively. We assume that the plate is infinitely long, so end effect can be neglected. The body force is ignored. The motion of the fluids is only due to oscillatory motion of the plate in its own plane. Therefore, in both the layers, fluid velocity field is one-dimensional, in the $x$-direction. The velocity fields for the lower and upper fluids have no dependence on the $x$ and $z$ coordinates. The velocity fields depend only on the $y$ coordinate and the time $t$. The velocity fields for the lower and upper fluids are governed by the reduced Navier–Stokes equations:

$$\frac{\partial u_1}{\partial t} = \nu_1 \frac{\partial^2 u_1}{\partial y^2}, \quad (2.1)$$

$$\frac{\partial u_2}{\partial t} = \nu_2 \frac{\partial^2 u_2}{\partial y^2}, \quad (2.2)$$

respectively, where $\nu_1$ is the kinematic viscosity of the lower fluid, and $\nu_2$ is that for the upper fluid. Note that the kinematic viscosity of a fluid is defined by $\nu = \frac{\mu}{\rho}$, where, $\rho$ is the fluid density, and $\mu$ is the dynamic viscosity or simply the viscosity of the fluid. In this work, we write $\mu_1$ and $\rho_1$ for dynamic viscosity and density of the lower fluid, and $\mu_2$ and $\rho_2$ for the corresponding quantities of the upper fluid. In Eqs. (2.1) and (2.2), $u_1(y,t)$ and $u_2(y,t)$ are velocities in the $x$-direction. Here, we consider no-slip boundary condition at the plate. Further, we consider continuity of velocity and that of shear stress at the interface between the fluids. Accordingly, the associated initial and boundary conditions are:

$$u_1(0,t) = U_0\cos(\omega t) \quad \text{or} \quad u_1(0,t) = U_0\sin(\omega t) \quad \text{for } t>0 \tag{2.3, 2.4}$$

$$u_1(y,0) = 0, \tag{2.5}$$

$$u_2(y,0) = 0, \tag{2.6}$$

$$u_1(h,t) = u_2(h,t), \tag{2.7}$$

$$\mu_1 \frac{\partial u_1}{\partial y}\bigg|_{y=h} = \mu_2 \frac{\partial u_2}{\partial y}\bigg|_{y=h}, \tag{2.8}$$

$$u_2(y\to\infty,t) = 0. \tag{2.9}$$

### 2.2. Solution

#### 2.2.1. Solution for the cosine oscillations of the plate

*2.2.1.1. Calculation of velocity fields.* In order to obtain the velocity fields for the lower and upper fluids, we need to find solution to the initial-boundary value problem consists of governing Eqs. (2.1) and (2.2) and initial and boundary conditions (2.3), and (2.5)–(2.9).

We employ the Laplace transform method to solve the mathematical problem. The method provides complete, analytical solution to a initial-boundary value problem that is valid for small and large values of time $t$. The Laplace transform of a given function $u(y,t)$ is defined by

$$\mathscr{L}(u(y,t)) = \tilde{u}(y,s) = \int_0^\infty u\exp(-st)dt, \tag{2.10}$$

where $s$ is the transform variable and $\exp(-st)$ is the kernel of the transform. For time $t>0$, the transform may be inverted using the following inversion formula[51]:

$$u(y,t) = \mathscr{L}^{-1}(\tilde{u}(y,s)) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\tilde{u}(y,s)\exp(st)ds, \tag{2.11}$$

where $\gamma$ is an arbitrary constant, and it must be greater than the real part of each of the singularities of $\tilde{u}(y,s)$. The formula is known as the Bromwich inversion integral.

We take the Laplace transforms of Eqs. (2.1) and (2.2), yielding

$$\tilde{u_1}'' - \frac{s}{\nu_1}\tilde{u_1} = 0, \tag{2.12}$$

$$\tilde{u_2}'' - \frac{s}{\nu_2}\tilde{u_2} = 0, \tag{2.13}$$

respectively. Note that the initial conditions (2.5) and (2.6) have been utilized to obtain the transforms. Here, primes stand for differentiation with respect to $y$. The transformations of the boundary conditions (2.3), and (2.7)–(2.9) result in

$$\tilde{u_1}(0,s) = U_0\frac{s}{s^2+\omega^2}, \tag{2.14}$$

$$\tilde{u_1}(h,s) = \tilde{u_2}(h,s), \tag{2.15}$$

$$\mu_1\tilde{u_1}'(h,s) = \mu_2\tilde{u_2}'(h,s), \tag{2.16}$$

$$\tilde{u_2}(y\to\infty,s) = 0, \tag{2.17}$$

respectively. The solutions of Eqs. (2.12) and (2.13) subject to boundary conditions (2.14)–(2.17) are

$$\tilde{u_1}(y,s) = \frac{sU_0}{(s^2+\omega^2)}\left[-\sum_{m=1}^{\infty}M^m\exp\left(-a_1\sqrt{s}\right) + \sum_{m=0}^{\infty}M^m\exp\left(-a_2\sqrt{s}\right)\right], \tag{2.18}$$

$$\tilde{u_2}(y,s) = \frac{sU_0}{(s^2+\omega^2)}\left[\sum_{m=0}^{\infty}(1-M)M^m\exp\left(-a_3\sqrt{s}\right)\right], \tag{2.19}$$

respectively, where

$$M = \frac{\alpha-1}{\alpha+1}, \tag{2.20}$$

$$a_1 = \frac{(2mh-y)}{\sqrt{\nu_1}}, \tag{2.21}$$

$$a_2 = \frac{(2mh+y)}{\sqrt{\nu_1}}, \tag{2.22}$$

$$a_3 = \left(\frac{(y-h)\sqrt{\nu_1} + (2m+1)h\sqrt{\nu_2}}{\sqrt{\nu_1\nu_2}}\right), \tag{2.23}$$

with

$$\alpha = \frac{\mu_2}{\mu_1}\sqrt{\frac{\nu_1}{\nu_2}}. \tag{2.24}$$

The Laplace transforms (2.18) and (2.19) can be inverted to obtain the velocity fields for the lower and upper fluids, $u_1(y,t)$ and $u_2(y,t)$, respectively. The velocity fields for the lower and upper fluids are

$$\begin{aligned} u_1(y,t) = \; & U_0\Bigg[-\sum_{m=1}^{\infty}M^m\exp(-a_1\sqrt{\tfrac{\omega}{2}})\cos(\omega t - a_1\sqrt{\tfrac{\omega}{2}}) \\ & + \sum_{m=0}^{\infty}M^m\exp(-a_2\sqrt{\tfrac{\omega}{2}})\cos(\omega t - a_2\sqrt{\tfrac{\omega}{2}})\Bigg] \\ & + \left\{-\frac{U_0}{\pi}\int_0^\infty \frac{-\sum_{m=1}^{\infty}M^m\sigma\exp(-\sigma t)\sin(a_1\sqrt{\sigma}) + \sum_{m=0}^{\infty}M^m\sigma\exp(-\sigma t)\sin(a_2\sqrt{\sigma})}{\sigma^2+\omega^2}d\sigma\right\}, \end{aligned} \tag{2.25}$$

$$\begin{aligned} u_2(y,t) = \; & U_0\left[\sum_{m=0}^{\infty}(1-M)M^m\exp(-a_3\sqrt{\tfrac{\omega}{2}})\cos(\omega t - a_3\sqrt{\tfrac{\omega}{2}})\right] \\ & + \left\{-\frac{U_0}{\pi}\int_0^\infty \frac{\sum_{m=0}^{\infty}(1-M)M^m\sigma\exp(-\sigma t)\sin(a_3\sqrt{\sigma})}{\sigma^2+\omega^2}d\sigma\right\}. \end{aligned} \tag{2.26}$$

Here, $M, a_1, a_2$, and $a_3$ are as defined in (2.20) and (2.21)–(2.23), respectively. We note that to invert the Laplace transforms (2.18) and (2.19) term by term, we have utilized the following result:

$$\begin{aligned} \mathscr{L}^{-1}\left(\frac{s\exp(-a\sqrt{s})}{s^2+\omega^2}\right) = \; & \exp(-a\sqrt{\frac{\omega}{2}})\cos(\omega t - a\sqrt{\frac{\omega}{2}}) \\ & - \frac{1}{\pi}\int_0^\infty \\ & \times \frac{\sigma\exp(-\sigma t)\sin(a\sqrt{\sigma})}{\sigma^2+\omega^2}d\sigma, \end{aligned} \tag{2.27}$$

where $a > 0$. The result is given in[52] as an exercise problem. In order to make this study as self-contained as possible, we have derived the result in detail in Appendix A.

If we let $t \to \infty$ into expression (2.25), the part within the curly brackets tends to zero. Therefore, the part within the curly brackets of the expression represents the transient velocity field for the lower fluid. And the remaining part of the expression corresponds to steady periodic velocity field for the fluid. Again, the part within the curly brackets of expression (2.26) represents transient velocity field for the upper fluid as it approaches zero as we let $t \to \infty$ into the expression. The remaining part of the expression corresponds to steady periodic velocity field for the fluid.

*2.2.1.2. Calculation of wall shear stress.* The velocity fields for both the lower and upper fluids have been determined. We now are interested in calculating wall shear stress. We know that the shear stress can be obtained using Newton's law of fluid friction:

$$\tau(y,t) = \mu \frac{\partial u(y,t)}{\partial y}, \tag{2.28}$$

where $\tau$ is the shear stress, $\mu$ is the viscosity (dynamic viscosity) of the fluid, and $u(y,t)$ is the velocity field.

The shear stress at the plate can be determined by putting $y = 0$ in the expression obtained by plugging expression (2.25) in the formula given by Eq. (2.28), which leads to

$$\begin{aligned}\tau_{1w}(0,t) = \mu_1 U_0 \Bigg[ &\sqrt{\frac{\omega}{\nu_1}} \Bigg( \sum_{m=1}^{\infty} M^m \exp\Big(-a_4\sqrt{\frac{\omega}{2}}\Big) \cos\Big(\omega t - a_4\sqrt{\frac{\omega}{2}} - \frac{3\pi}{4}\Big) \\ &+ \sum_{m=0}^{\infty} M^m \exp\Big(-a_4\sqrt{\frac{\omega}{2}}\Big) \cos\Big(\omega t - a_4\sqrt{\frac{\omega}{2}} - \frac{3\pi}{4}\Big) \Bigg) \Bigg] \\ &+ \left\{ -\frac{\mu_1 U_0}{\pi} \int_0^{\infty} \frac{\sqrt{\frac{\sigma}{\nu_1}} \left( \sum_{m=1}^{\infty} M^m \sigma \exp(-\sigma t) \cos(a_4\sqrt{\sigma}) + \sum_{m=0}^{\infty} M^m \sigma \exp(-\sigma t) \cos(a_4\sqrt{\sigma}) \right)}{\sigma^2 + \omega^2} d\sigma \right\},\end{aligned} \tag{2.29}$$

where $M$ is as defined in(2.20), and

$$a_4 = \frac{2mh}{\sqrt{\nu_1}}. \tag{2.30}$$

If we let $t \to \infty$ into the wall shear stress given by Eq. (2.29), the part within the curly brackets approaches zero. So the part inside the curly brackets represents the transient wall shear stress. And the remaining part of expression (2.29) corresponds to steady-state wall shear stress.

*2.2.1.3. Special Case: single-layer limit.* When $h \to \infty, h$ being the thickness of the lower fluid, the two-layer problem reduces to classical Stokes' second problem for a single-layer fluid with the cosine oscillations of the plate. If we let $h \to \infty, \mu_1 = \mu_2 = \mu$ (say the viscosity of the single-layer fluid), and $\nu_1 = \nu_2 = \nu$ (say the kinematic viscosity of the single-layer fluid) in the velocity field for the lower fluid, (2.25), we find that each of the terms of the series in the expression becomes identically zero, except those that we get for $m = 0$. Thus, for the case the velocity field for the lower fluid, (2.25), becomes

$$\begin{aligned}u_{cla}(y,t) = U_0 \Bigg[ &\exp\Big(-y\sqrt{\frac{\omega}{2\nu}}\Big) \cos\Big(\omega t - y\sqrt{\frac{\omega}{2\nu}}\Big) \\ &- \frac{1}{\pi} \int_0^{\infty} \frac{\sigma \exp(-\sigma t) \sin(y\sqrt{\frac{\sigma}{\nu}})}{\sigma^2 + \omega^2} d\sigma \Bigg].\end{aligned} \tag{2.31}$$

Expression (2.31) is the velocity field for classical Stokes' second problem when the plate oscillates as $U_0 \cos(\omega t)$. The velocity field (2.31) agrees with the result for the flow that can be easily obtained from the related result reported in [4].

Note that the preceding velocity field for a single-layer fluid can also be deduced from the velocity field for the upper fluid, (2.26), as a special case. If we let $h = 0$ (meaning that the lower fluid ceases to exist), $\mu_1 = \mu_2 = \mu$ (say the viscosity of the single-layer fluid), and $\nu_1 = \nu_2 = \nu$ (say the kinematic viscosity of the single-layer fluid) in expression (2.26), we obtain the result.

*2.2.2. Solution for the sine oscillations of the plate*

*2.2.2.1. Calculation of velocity fields.* The velocity fields for the lower and upper fluids can be determined by solving the initial-boundary value problem consists of governing Eqs. (2.1) and (2.2) and initial and boundary conditions (2.4)–(2.5)–(2.9). We note that the initial-boundary value problem is the same as the one we have dealt earlier in this section, except that condition (2.4) replaces condition (2.3). Therefore, to solve the mathematical problem in hand, we follow the same procedure adopted earlier. We obtain the velocity fields for the lower and upper fluids as follows:

$$\begin{aligned}u_1(y,t) = U_0 \Bigg[ &-\sum_{m=1}^{\infty} M^m \exp\Big(-a_1\sqrt{\tfrac{\omega}{2}}\Big) \sin\Big(\omega t - a_1\sqrt{\tfrac{\omega}{2}}\Big) \\ &+ \sum_{m=0}^{\infty} M^m \exp\Big(-a_2\sqrt{\tfrac{\omega}{2}}\Big) \sin\Big(\omega t - a_2\sqrt{\tfrac{\omega}{2}}\Big) \Bigg] \\ &+ \left\{ \frac{U_0\omega}{\pi} \int_0^{\infty} \frac{-\sum_{m=1}^{\infty} M^m \exp(-\sigma t) \sin(a_1\sqrt{\sigma}) + \sum_{m=0}^{\infty} M^m \exp(-\sigma t) \sin(a_2\sqrt{\sigma})}{\sigma^2+\omega^2} d\sigma \right\},\end{aligned} \tag{2.32}$$

$$\begin{aligned}u_2(y,t) = \quad U_0 \Bigg[ &\sum_{m=0}^{\infty} (1-M) M^m \exp\Big(-a_3\sqrt{\tfrac{\omega}{2}}\Big) \sin\Big(\omega t - a_3\sqrt{\tfrac{\omega}{2}}\Big) \Bigg] \\ &+ \left\{ \frac{U_0\omega}{\pi} \int_0^{\infty} \frac{\sum_{m=0}^{\infty} (1-M) M^m \exp(-\sigma t) \sin(a_3\sqrt{\sigma})}{\sigma^2+\omega^2} d\sigma \right\},\end{aligned} \tag{2.33}$$

respectively. Here, $M, a_1, a_2$, and $a_3$ are as defined in (2.20) and (2.21)–(2.23).

It is to be noted here that in expression (2.32), the part within the curly brackets represents the transient velocity field for the lower fluid as it approaches zero as we let $t \to \infty$ into the expression. The remaining part of the expression represents the steady periodic velocity field for the fluid. Similarly, in expression (2.33), the part inside the curly brackets represents the transient velocity field for the upper fluid, and the remaining part of the expression represents the steady periodic velocity field for the fluid.

It is worth mentioning that Duffy[53] has solved the initial-boundary value problem that we have tackled here, obtaining mathematical solution similar to the one reported here. He has indicated that the solution can be used to investigate the physical problem concerning heat conduction in a two-layer solid body.

*2.2.2.2. Calculation of wall shear stress.* We have calculated the velocity fields for the lower and upper fluids. We now intend to evaluate the shear stress at the plate. The shear stress at the plate can be found by substituting $y = 0$ into the expression obtained by plugging expression (2.32) in the formula given by Eq. (2.28), which leads to

$$\tau_{1w}(0,t) = \mu_1 U_0 \left[ \sqrt{\tfrac{\omega}{\nu_1}} \left( \sum_{m=1}^{\infty} M^m \exp(-a_4\sqrt{\tfrac{\omega}{2}}) \sin(\omega t - a_4\sqrt{\tfrac{\omega}{2}} - \tfrac{3\pi}{4}) + \sum_{m=0}^{\infty} M^m \exp(-a_4\sqrt{\tfrac{\omega}{2}}) \sin(\omega t - a_4\sqrt{\tfrac{\omega}{2}} - \tfrac{3\pi}{4}) \right) \right] + \left\{ \mu_1 U_0 \tfrac{\omega}{\pi} \int_0^{\infty} \frac{\sqrt{\tfrac{\sigma}{\nu_1}} \left( \sum_{m=1}^{\infty} M^m \exp(-\sigma t)\cos(a_4\sqrt{\sigma}) + \sum_{m=0}^{\infty} M^m \exp(-\sigma t)\cos(a_4\sqrt{\sigma}) \right)}{\sigma^2+\omega^2} d\sigma \right\}, \tag{2.34}$$

where $M$ and $a_4$ are as defined in (2.20) and (2.30), respectively. Here, the part within the curly brackets represents transient shear stress at the plate as it tends to zero as we let $t \to \infty$ into the result for wall shear stress. The remaining part of the result represents steady-state shear stress at the plate. It is valid for large values of time $t$.

*2.2.2.3. Special Case: single-layer limit.* Earlier in this section, we have deduced the velocity field for Stokes' second problem for a single-layer fluid, (2.31), as a special case. The velocity field corresponds to the case where the plate oscillates as $U_0\cos(\omega t)$ (the cosine oscillations). We follow the same procedure to deduce the velocity field for Stokes' second problem for a single-layer fluid related to the sine oscillations of the plate from the velocity field for the lower fluid, (2.32). The result is

$$u_{cla}(y,t) = U_0 \left[ \exp(-y\sqrt{\frac{\omega}{2\nu}}) \sin(\omega t - y\sqrt{\frac{\omega}{2\nu}}) + \frac{\omega}{\pi} \int_0^{\infty} \frac{\exp(-\sigma t)\sin(y\sqrt{\frac{\sigma}{\nu}})}{\sigma^2+\omega^2} d\sigma \right]. \tag{2.35}$$

The velocity field (2.35) agrees with the result for the flow that can be easily obtained from the related result reported in [4].

Note that the preceding velocity field for a single-layer fluid can also be deduced from the velocity field for the upper fluid, (2.33), by adopting the procedure outlined earlier in this section.

## 3. Oscillatory Couette flow for a two-layer fluid

### 3.1. Mathematical Formulation

Consider two superposed layers of two incompressible, viscous fluids over a flat plate that coincides with the $x$-$z$ plane of the Cartesian coordinate system $(x, y, z)$. Suppose that the two-layer fluid is bounded above by another plate which is parallel to the lower plate. Suppose that the plates are at a distance $H$ apart. The $y$-axis is the co-ordinate normal to the plates. Both of the two fluids that form the layers are Newtonian. The fluids are of different viscosities, densities, and thicknesses. The two fluids of the layers are immiscible. We suppose that the lower fluid fills the region $0 \leqslant y \leqslant h$, $h$ being a positive real number. And the upper fluid occupies the region $h \leqslant y \leqslant H$. The fluids and the plates are initially at rest and then the lower plate starts to oscillate in its own plane along $x$-axis with velocity $U_0\cos(\omega t)$ or $U_0\sin(\omega t)$, where $U_0, \omega$, and $t$ being the plate velocity amplitude, frequency of oscillations, and time, respectively. It is considered that the plates are infinitely long, so end effect can be neglected. The body force is ignored. The motion of the fluids are caused only by the oscillatory motion of the lower plate in its own plane. Therefore, in both the layers, fluid velocity field is one-dimensional, in the $x$-direction. The velocity fields for the lower and upper fluids have no dependence on the $x$ and $z$ coordinates. The velocity fields depend only on the $y$ coordinate and the time $t$. The velocity fields for the lower and upper fluids are governed by the reduced Navier–Stokes Eqs. (2.1) and (2.2), respectively. We consider the no-slip boundary condition at the plates. Besides, we consider continuity of velocity and that of shear stress at the interface between the fluids. Accordingly, the boundary and initial conditions are

$$u_1(0,t) = U_0\cos(\omega t) \quad \text{or} \quad u_1(0,t) = U_0\sin(\omega t) \quad \text{for} \quad t > 0 \tag{3.1, 3.2}$$

$$u_1(y,0) = 0 \tag{3.3}$$

$$u_2(y,0) = 0 \tag{3.4}$$

$$u_1(h,t) = u_2(h,t) \tag{3.5}$$

$$\mu_1 \frac{\partial u_1}{\partial y}\bigg|_{y=h} = \mu_2 \frac{\partial u_2}{\partial y}\bigg|_{y=h} \tag{3.6}$$

$$u_2(H,t) = 0 \tag{3.7}$$

### 3.2. Solution

#### 3.2.1. Solution for the cosine oscillations of the plate

*3.2.1.1. Calculation of velocity fields.* In order to determine the velocity fields for the lower and upper fluids, we need to obtain solution to the initial-boundary value problem consists of governing Eqs. (2.1) and (2.2), and initial and boundary conditions (3.1), and (3.3)–(3.7). The Laplace transforms of Eqs. (2.1) and (2.2) are Eqs. (2.12) and (2.13), respectively. We note that initial conditions (3.3) and (3.4) have been utilized to obtain the transforms. Again, the transforms of the boundary conditions (3.1), (3.5), and (3.6) are Eqs. (2.14)–(2.16), respectively. And the transform of the boundary condition (3.7) is

$$\tilde{u_2}(H,s) = 0. \tag{3.8}$$

The solutions of Eqs. (2.12) and (2.13) subject to boundary conditions (2.14)–(2.16) and (3.8) are

$$\tilde{u_1}(y,s) = -\frac{U_0 s}{s^2+\omega^2} \left[ \frac{\{\sinh(p)\sinh(r-q) + \alpha\cosh(p)\cosh(r-q)\}\sinh(\sqrt{\frac{s}{\nu_1}}y)}{F_1(s)} - \cosh(\sqrt{\frac{s}{\nu_1}}y) \right], \tag{3.9}$$

$$\tilde{u_2}(y,s) = \frac{U_0 s}{s^2+\omega^2} \left[ \frac{\sinh(r - \sqrt{\frac{s}{\nu_2}}y)}{F_1(s)} \right], \tag{3.10}$$

respectively, where

$$p = \sqrt{\frac{s}{\nu_1}}h, \tag{3.11}$$

$$q = \sqrt{\frac{s}{\nu_2}}h, \tag{3.12}$$

$$r = \sqrt{\frac{s}{\nu_2}}H, \tag{3.13}$$

$$F_1(s) = \cosh(p)\sinh(r-q) + \alpha\sinh(p)\cosh(r-q), \quad (3.14)$$

and $\alpha$ is as defined in (2.24).

Each of Eqs. (3.9) and (3.10) has simple poles at $s = i\omega$ and $s = -i\omega$. Also, each of these equations has infinite number of poles which lie on the negative real axis at $s = -k_m^2$, where $k_m$ is a real number and $m$ is the index number (an integer) of the pole. Here, $k_m$ can be obtained from the following equation:

$$\alpha\tan\left(k_m\frac{h}{\sqrt{\nu_1}}\right) = -\tan\left(k_m\frac{(H-h)}{\sqrt{\nu_2}}\right). \quad (3.15)$$

The steady periodic velocity fields for the lower and upper fluids are related to the simple poles at $s = i\omega$ and $s = -i\omega$, whereas the poles located at $s = -k_m^2$ are responsible for the transient velocity fields.

The Laplace inverse for $\tilde{u}_1(y,s)$, (3.9), and $\tilde{u}_2(y,s)$, (3.10), can be computed, respectively, from the following relations [4]:

$$u_1(y,t) = \sum_{m=1}^{\infty} Res[\tilde{u}_1(y,s)]_{-k_m^2} + Res[\tilde{u}_1(y,s)]_{i\omega} + Res[\tilde{u}_1(y,s)]_{-i\omega}, \quad (3.16)$$

$$u_2(y,t) = \sum_{m=1}^{\infty} Res[\tilde{u}_2(y,s)]_{-k_m^2} + Res[\tilde{u}_2(y,s)]_{i\omega} + Res[\tilde{u}_2(y,s)]_{-i\omega}, \quad (3.17)$$

where $Res$ stands for the residue.

The steady periodic velocity field for the lower fluid can be found by evaluating residues at $s = i\omega$ and $s = -i\omega$, which yields

$$u_{1s}(y,t) = \frac{U_0}{A^2+B^2}\Big[[-(g_1(y)A + g_2(y)B) + \cosh\left(\sqrt{\frac{\omega}{2\nu_1}}y\right)\cos\left(\sqrt{\frac{\omega}{2\nu_1}}y\right)(A^2+B^2)]\cos(\omega t) - [g_1(y)B - g_2(y)A + \sinh\left(\sqrt{\frac{\omega}{2\nu_1}}y\right)\sin\left(\sqrt{\frac{\omega}{2\nu_1}}y\right)(A^2+B^2)]\sin(\omega t)\Big]. \quad (3.18)$$

In a similar way the steady periodic velocity field for the upper fluid can be determined, which is

$$u_{2s}(y,t) = \frac{U_0}{A^2+B^2}[(g_3(y)A + g_4(y)B)\cos(\omega t) - (g_4(y)A - g_3(y)B)\sin(\omega t)]. \quad (3.19)$$

In expressions (3.18) and (3.19), the constants $A$ and $B$ are defined as follows:

$$\begin{aligned} A = {} & \alpha\cos(a)\sinh(a)\cos(b-c)\cosh(b-c) \\ & - \alpha\sin(a)\cosh(a)\sin(b-c)\sinh(b-c) \\ & - \cos(a)\cosh(a)\cos(b-c)\sinh(b-c) \\ & + \sin(a)\sinh(a)\sin(b-c)\cosh(b-c), \end{aligned} \quad (3.20)$$

$$\begin{aligned} B = {} & \alpha\cos(a)\sinh(a)\sin(b-c)\sinh(b-c) + \alpha\sin(a) \\ & \times\cosh(a)\cos(b-c)\cosh(b-c) - \sin(a)\sinh(a) \\ & \times\cos(b-c)\sinh(b-c) - \cos(a)\cosh(a)\sin(b \\ & - c)\cosh(b-c), \end{aligned} \quad (3.21)$$

with

$$a = \sqrt{\frac{\omega}{2\nu_1}}h, \quad (3.22)$$

$$b = \sqrt{\frac{\omega}{2\nu_2}}h, \quad (3.23)$$

$$c = \sqrt{\frac{\omega}{2\nu_2}}H, \quad (3.24)$$

and $\alpha$ is as defined in Eq. (2.24).

The functions $g_1(y), g_2(y), g_3(y)$, and $g_4(y)$ are defined as follows:

$$\begin{aligned} g_1(y) = {} & -\sin(e)\cosh(e)[\alpha\sin(a)\sinh(a)\cos(b-c)\cosh(b-c) \\ & + \alpha\cos(a)\cosh(a)\sin(b-c)\sinh(b-c) \\ & - \cos(a)\sinh(a)\sin(b-c)\cosh(b-c) \\ & - \sin(a)\cosh(a)\cos(b-c)\sinh(b-c)] \\ & + \cos(e)\sinh(e)[\alpha\cos(a)\cosh(a)\cos(b-c) \\ & \cosh(b-c) - \alpha\sin(a)\sinh(a)\sin(b-c)\sinh(b-c) \\ & - \cos(a)\sinh(a)\cos(b-c)\sinh(b-c) \\ & + \sin(a)\cosh(a)\sin(b-c)\cosh(b-c)], \end{aligned} \quad (3.25)$$

$$\begin{aligned} g_2(y) = {} & \sin(e)\cosh(e)[\alpha\cos(a)\cosh(a)\cos(b-c)\cosh(b-c) \\ & - \alpha\sin(a)\sinh(a)\sin(b-c)\sinh(b-c) \\ & - \cos(a)\sinh(a)\cos(b-c)\sinh(b-c) \\ & + \sin(a)\cosh(a)\sin(b-c)\cosh(b-c)] \\ & + \cos(e)\sinh(e)[\alpha\sin(a)\sinh(a)\cos(b-c)\cosh(b-c) \\ & + \alpha\cos(a)\cosh(a)\sin(b-c)\sinh(b-c) \\ & - \cos(a)\sinh(a)\sin(b-c)\cosh(b-c) \\ & - \sin(a)\cosh(a)\cos(b-c)\sinh(b-c)], \end{aligned} \quad (3.26)$$

$$g_3(y) = \cos(c-d)\sinh(c-d), \quad (3.27)$$

$$g_4(y) = \cosh(c-d)\sin(c-d), \quad (3.28)$$

with

$$d = \sqrt{\frac{\omega}{2\nu_2}}y, \quad (3.29)$$

$$e = \sqrt{\frac{\omega}{2\nu_1}}y. \quad (3.30)$$

Again, the transient velocity field for the lower fluid can be obtained by calculating the residues at all $s = -k_m^2$, which results in

$$u_{1t}(y,t) = -\sum_{m=1}^{\infty}\frac{2U_0k_m^3}{k_m^4+\omega^2}\left[\frac{F_2(k_m)\sin(k_m\frac{y}{\sqrt{\nu_1}}) - F_3(k_m)\cos(k_m\frac{y}{\sqrt{\nu_1}})}{F_4(k_m)}\right]\exp(-k_m^2 t). \quad (3.31)$$

We follow the similar procedure to determine the transient velocity field for the upper fluid, yielding

$$u_{2t}(y,t) = \sum_{m=1}^{\infty}\frac{2U_0k_m^3}{k_m^4+\omega^2}\left[\frac{\sin(k_m\frac{(H-y)}{\sqrt{\nu_2}})}{F_4(k_m)}\right]\exp(-k_m^2 t). \quad (3.32)$$

In expressions (3.31) and (3.32),

$$\begin{aligned} F_2(k_m) = {} & -\sin\left(k_m\frac{h}{\sqrt{\nu_1}}\right)\sin\left(k_m\frac{(H-h)}{\sqrt{\nu_2}}\right) \\ & + \alpha\cos\left(k_m\frac{h}{\sqrt{\nu_1}}\right)\cos\left(k_m\frac{(H-h)}{\sqrt{\nu_2}}\right), \end{aligned} \quad (3.33)$$

$$\begin{aligned} F_3(k_m) = {} & \cos\left(k_m\frac{h}{\sqrt{\nu_1}}\right)\sin\left(k_m\frac{(H-h)}{\sqrt{\nu_2}}\right) \\ & + \alpha\sin\left(k_m\frac{h}{\sqrt{\nu_1}}\right)\cos\left(k_m\frac{(H-h)}{\sqrt{\nu_2}}\right), \end{aligned} \quad (3.34)$$

$$F_4(k_m) = \Bigg[\frac{(H-h)}{\sqrt{v_2}}\Bigg\{\cos(k_m\frac{h}{\sqrt{v_1}})\cos(k_m\frac{(H-h)}{\sqrt{v_2}}) -\alpha\sin(k_m\frac{h}{\sqrt{v_1}})\sin(k_m\frac{(H-h)}{\sqrt{v_2}})\Bigg\} + \frac{h}{\sqrt{v_1}}\Bigg\{-\sin(k_m\frac{h}{\sqrt{v_1}})\sin(k_m\frac{(H-h)}{\sqrt{v_2}}) +\alpha\cos(k_m\frac{h}{\sqrt{v_1}})\cos(k_m\frac{(H-h)}{\sqrt{v_2}})\Bigg\}\Bigg]. \quad (3.35)$$

Now, in accordance with (3.16) and (3.17), the complete velocity field for each of the lower and upper fluids is the sum of the respective steady periodic and transient velocity fields. Therefore, the complete velocity fields for the lower and the upper fluids are

$$u_1(y,t) = u_{1s}(y,t) + u_{1t}(y,t), \quad (3.36)$$

$$u_2(y,t) = u_{2s}(y,t) + u_{2t}(y,t), \quad (3.37)$$

respectively. Here, $u_{1s}(y,t), u_{1t}(y,t), u_{2s}(y,t)$, and $u_{2t}(y,t)$ are given by (3.18), (3.31), (3.19), and (3.32), respectively.

*3.2.1.2. Calculation of wall shear stresses.* The velocity fields for both the lower and upper fluids have been explicitly obtained. We now are interested in evaluating the shear stresses at the plates. The steady-state and transient shear stresses in the lower fluid can be calculated by plugging the expressions (3.18) and (3.31) in the formula given by the Eq. (2.28), respectively, obtaining

$$\tau_{1s}(y,t) = \frac{\mu_1 U_0}{A^2+B^2}\Big[\cos(\omega t)[-(g_1'(y)A + g_2'(y)B) + \sqrt{\frac{\omega}{2v_1}}(\sinh(e)\cos(e) - \sin(e)\cosh(e))(A^2+B^2)] - \sin(\omega t)[g_1'(y)B - g_2'(y)A + \sqrt{\frac{\omega}{2v_1}}(\cosh(e)\sin(e) + \sinh(e)\cos(e))(A^2+B^2)]\Big], \quad (3.38)$$

$$\tau_{1t}(y,t) = -\sum_{m=1}^{\infty}\Big(\frac{2\mu_1 U_0 k_m^4}{\sqrt{v_1}(k_m^4+\omega^2)}\Big) \times \left[\frac{F_2(k_m)\cos(k_m\frac{y}{\sqrt{v_1}}) + F_3(k_m)\sin(k_m\frac{y}{\sqrt{v_1}})}{F_4(k_m)}\right]\exp(-k_m^2 t), \quad (3.39)$$

respectively. Here, the constants $A$ and $B$ are as defined in (3.20) and (3.21). The functions $g_1(y)$ and $g_2(y)$ are as defined in (3.25) and (3.26). And $e$ is as defined in (3.30). Also, $F_2(k_m), F_3(k_m)$, and $F_4(k_m)$ are as defined in (3.33)–(3.35), respectively. Note that here primes denote differentiation with respect to $y$.

Again, we can evaluate the steady-state and transient shear stresses in the upper fluid by plugging, respectively, the expressions (3.19) and (3.32) in the formula given by Eq. (2.28), yielding

$$\tau_{2s}(y,t) = \frac{\mu_2 U_0}{A^2+B^2}\left[(g_3'(y)A + g_4'(y)B)\cos(\omega t) - (g_4'(y)A - g_3'(y)B)\sin(\omega t)\right], \quad (3.40)$$

$$\tau_{2t}(y,t) = -\sum_{m=1}^{\infty}\frac{2\mu_2 U_0 k_m^4}{\sqrt{v_2}(k_m^4+\omega^2)}\left[\frac{\cos(k_m\frac{(H-y)}{\sqrt{v_2}})}{F_4(k_m)}\right]\exp(-k_m^2 t), \quad (3.41)$$

respectively. Here, the functions $g_3(y)$ and $g_4(y)$ are as defined in (3.27) and (3.28).

The steady periodic and transient shear stresses at the oscillating plate can be evaluated by substituting $y = 0$ into expressions (3.38) and (3.39), respectively, obtaining

$$\tau_{1ws}(0,t) = \frac{\mu_1 U_0}{A^2+B^2}\sqrt{\frac{\omega}{2v_1}}[\cos(\omega t)[-((-K_1+K_2)A + (K_1+K_2)B)] - \sin(\omega t)[(-K_1+K_2)B - (K_1+K_2)A]], \quad (3.42)$$

$$\tau_{1wt}(0,t) = -\sum_{m=1}^{\infty}\Big(\frac{2\mu_1 U_0 k_m^4}{\sqrt{v_1}(k_m^4+\omega^2)}\Big)\left[\frac{F_2(k_m)}{F_4(k_m)}\right]\exp(-k_m^2 t), \quad (3.43)$$

respectively. Here, the constants $K_1$ and $K_2$ are defined as follows:

$$K_1 = \alpha\sin(a)\sinh(a)\cos(b-c)\cosh(b-c) +\alpha\cos(a)\cosh(a)\sin(b-c)\sinh(b-c) - \cos(a)\sinh(a)\sin(b-c)\cosh(b-c) - \sin(a)\cosh(a)\cos(b-c)\sinh(b-c), \quad (3.44)$$

$$K_2 = \alpha\cos(a)\cosh(a)\cos(b-c)\cosh(b-c) -\alpha\sin(a)\sinh(a)\sin(b-c)\sinh(b-c) - \cos(a)\sinh(a)\cos(b-c)\sinh(b-c) + \sin(a)\cosh(a)\sin(b-c)\cosh(b-c), \quad (3.45)$$

where $\alpha, a, b$, and $c$ are as defined in Eqs. (2.24) and (3.22)–(3.24), respectively.

Again, the steady periodic and transient shear stresses at the stationary plate can be calculated by putting $y = H$ in expressions (3.40) and (3.41), respectively, yielding

$$\tau_{2ws}(H,t) = -\frac{\mu_2 U_0}{A^2+B^2}\sqrt{\frac{\omega}{2v_2}}[(A+B)\cos(\omega t) - (A-B)\sin(\omega t)], \quad (3.46)$$

$$\tau_{2wt}(H,t) = -\sum_{m=1}^{\infty}\frac{2\mu_2 U_0 k_m^4}{\sqrt{v_2}(k_m^4+\omega^2)}\left[\frac{\exp(-k_m^2 t)}{F_4(k_m)}\right], \quad (3.47)$$

respectively.

*3.2.1.3. Special case: single-layer limit.* The two-layer fluid flow problem reduces to a single-layer one when $h$, the thickness of the lower fluid, becomes equal to $H$ or zero. We note that $H$ is the distance between the plates. If we let $h = H, \mu_1 = \mu_2 = \mu$ (say the viscosity of the single-layer fluid), and $v_1 = v_2 = v$ (say the kinematic viscosity of the single-layer fluid) in the steady periodic velocity field for lower fluid, (3.18), we obtain steady-state velocity field for the single-layer fluid flow. Similarly, the transient velocity field for the single-layer fluid flow can be obtained from expression (3.31). The results for the steady-periodic and transient velocity fields are

$$u_{cssc}(y,t) = \frac{U_0}{A_1^2+B_1^2}[(f_1(y)A_1 + f_2(y)B_1)\cos(\omega t) - (f_2(y)A_1 - f_1(y)B_1)\sin(\omega t)], \quad (3.48)$$

$$u_{ctsc}(y,t) = \sum_{m=1}^{\infty}\frac{2U_0 k_m^3\sqrt{v}}{k_m^4+\omega^2}\left[\frac{\sin(k_m\frac{(H-y)}{\sqrt{v}})}{H\cos(k_m\frac{(H)}{\sqrt{v}})}\right]\exp(-k_m^2 t), \quad (3.49)$$

respectively. Here, the functions $f_1(y)$ and $f_2(y)$ are defined as follows:

$$f_1(y) = \cos\Big(\sqrt{\frac{\omega}{2v}}(H-y)\Big)\sinh\Big(\sqrt{\frac{\omega}{2v}}(H-y)\Big), \quad (3.50)$$

$$f_2(y) = \cosh\Big(\sqrt{\frac{\omega}{2v}}(H-y)\Big)\sin\Big(\sqrt{\frac{\omega}{2v}}(H-y)\Big). \quad (3.51)$$

The constants $A_1$ and $B_1$ are defined as the following:

$$A_1 = \cos\left(\sqrt{\frac{\omega}{2\nu}}H\right)\sinh\left(\sqrt{\frac{\omega}{2\nu}}H\right), \tag{3.52}$$

$$B_1 = \sin\left(\sqrt{\frac{\omega}{2\nu}}H\right)\cosh\left(\sqrt{\frac{\omega}{2\nu}}H\right). \tag{3.53}$$

We note that $k_m$ can be found by the following relation:

$$k_m = m\frac{\sqrt{\nu}}{H}\pi, \quad m = 1, 2, 3, \ldots, \tag{3.54}$$

which is deduced from Eq. (3.15).

The complete velocity field for the single-layer fluid is the sum of the steady-state velocity field, (3.48) and the transient velocity field, (3.49). The transient dies out as the time $t$ progresses.

We note that some hints on the above deductions for the single-layer fluid are given in Appendix B, which a reader might find helpful.

Note that the steady-state and transient velocity fields for a single-layer flow, (3.48) and (3.49), can also be deduced from steady-state and transient velocity fields for the upper fluid, (3.19) and (3.32), respectively. In order to obtain the results, we need to let $h = 0$(meaning that the lower fluid ceases to exist), $\mu_1 = \mu_2 = \mu$ (say the viscosity of the single-layer fluid), and $\nu_1 = \nu_2 = \nu$ (say the kinematic viscosity of the single-layer fluid) in expressions (3.19) and (3.32). Some helpful hints about the deductions are given in Appendix B.

It is important to be noted here that the complete velocity field for the single-layer fluid found here as a special case has not, so far as we are aware, been previously reported in the literature.

We can now compute the shear stresses at the oscillating and fixed plates for the single-layer fluid. We can calculate steady-state and transient shear stresses at the oscillating plate by substituting $y = 0$ into the expressions obtained by plugging expressions (3.48) and (3.49) in formula (2.28), respectively. Again, if we put $y = H$ in the expressions, it will result in steady periodic and transient shear stresses at the fixed plate. The steady periodic and transient shear stresses at the oscillating plate are

$$\tau_{cssc}(0,t) = \frac{\mu U_0}{A_1^2 + B_1^2}[(A_2A_1 + B_2B_1)\cos(\omega t) - (B_2A_1 - A_2B_1)\sin(\omega t)], \tag{3.55}$$

$$\tau_{ctsc}(0,t) = -\sum_{m=1}^{\infty}\frac{2\mu U_0 k_m^4}{H(k_m^4 + \omega^2)}\exp(-k_m^2 t), \tag{3.56}$$

respectively. The constants $A_2$ and $B_2$ are defined as follows:

$$A_2 = \sqrt{\frac{\omega}{2\nu}}\left[\sin\left(\sqrt{\frac{\omega}{2\nu}}H\right)\sinh\left(\sqrt{\frac{\omega}{2\nu}}H\right) - \cos\left(\sqrt{\frac{\omega}{2\nu}}H\right)\cosh\left(\sqrt{\frac{\omega}{2\nu}}H\right)\right], \tag{3.57}$$

$$B_2 = -\sqrt{\frac{\omega}{2\nu}}\left[\sin\left(\sqrt{\frac{\omega}{2\nu}}H\right)\sinh\left(\sqrt{\frac{\omega}{2\nu}}H\right) + \cos\left(\sqrt{\frac{\omega}{2\nu}}H\right)\cosh\left(\sqrt{\frac{\omega}{2\nu}}H\right)\right]. \tag{3.58}$$

Again, the steady-state and transient shear stresses at the stationary plate are

$$\tau_{cssc}(H,t) = -\frac{\mu U_0}{A_1^2 + B_1^2}\sqrt{\frac{\omega}{2\nu}}[(A_1 + B_1)\cos(\omega t) - (A_1 - B_1)\sin(\omega t)], \tag{3.59}$$

$$\tau_{ctsc}(H,t) = -\sum_{m=1}^{\infty}\frac{2\mu U_0 k_m^4}{k_m^4 + \omega^2}\left[\frac{1}{H\cos(k_m\frac{(H)}{\sqrt{\nu}})}\right]\exp(-k_m^2 t), \tag{3.60}$$

respectively.

*3.2.2. Solution for the sine oscillations of the plate*

*3.2.2.1. Calculation of velocity fields.* In order to obtain the velocity fields for the lower and upper fluids, we need to obtain solution to the initial-boundary value problem consists of governing Eqs. (2.1) and (2.2) and initial and boundary conditions (3.2)–(3.7). We note that the mathematical problem is the same as the one we have tackled earlier in this section, except that condition (3.2) replaces condition (3.1). Therefore, to deal with the initial-boundary value problem in hand, we adopt the same procedure that we have followed earlier.

The steady periodic and transient velocity fields for the lower fluid are

$$u_{1s}(y,t) = \frac{U_0}{A^2 + B^2}[[g_1(y)B - g_2(y)A + \sinh(e)\sin(e)(A^2 + B^2)]\cos(\omega t) + [-(g_1(y)A + g_2(y)B) + \cosh(e)\cos(e)(A^2 + B^2)]\sin(\omega t)], \tag{3.61}$$

$$u_{1t}(y,t) = \sum_{m=1}^{\infty}\frac{2U_0\omega k_m}{k_m^4 + \omega^2}\left[\frac{F_2(k_m)\sin(k_m\frac{y}{\sqrt{\nu_1}}) - F_3(k_m)\cos(k_m\frac{y}{\sqrt{\nu_1}})}{F_4(k_m)}\right]\exp(-k_m^2 t), \tag{3.62}$$

respectively. And the steady periodic and transient velocity fields for the upper fluid are

$$u_{2s}(y,t) = \frac{U_0}{A^2 + B^2}[(g_4(y)A - g_3(y)B)\cos(\omega t) + (g_3(y)A + g_4(y)B)\sin(\omega t)], \tag{3.63}$$

$$u_{2t}(y,t) = -\sum_{m=1}^{\infty}\frac{2U_0\omega k_m}{k_m^4 + \omega^2}\left[\frac{\sin(k_m\frac{(H-y)}{\sqrt{\nu_2}})}{F_4(k_m)}\right]\exp(-k_m^2 t), \tag{3.64}$$

respectively.

In the above expressions the constants $A$ and $B$ are as defined in Eqs. (3.20) and (3.21). The functions $g_1(y), g_2(y), g_3(y), g_4(y)$, and $e$ are as defined in Eqs. (3.25)–(3.28) and (3.30). Also $F_2(k_m), F_3(k_m)$, and $F_4(k_m)$ are as defined in Eqs. (3.33)–(3.35).

Note that the complete velocity field for each of the lower and upper fluids is the sum of the corresponding steady periodic and transient velocity fields. Accordingly, the complete velocity fields for the lower and upper fluids are

$$u_1(y,t) = u_{1s}(y,t) + u_{1t}(y,t), \tag{3.65}$$

$$u_2(y,t) = u_{2s}(y,t) + u_{2t}(y,t), \tag{3.66}$$

respectively. Here, $u_{1s}(y,t), u_{1t}(y,t), u_{2s}(y,t)$, and $u_{2t}(y,t)$ are given by Eqs. (3.61)–(3.64), respectively.

*3.2.2.2. Calculation of wall shear stresses.* As the velocity fields for both the lower and upper fluids have been obtained, we can now compute the shear stresses at the oscillating and fixed plates. The steady periodic and transient shear stresses in the lower fluid can be found by substituting expressions (3.61) and (3.62) into formula (2.28), respectively. The results are

$$\tau_{1s}(y,t) = \frac{\mu_1 U_0}{A^2 + B^2}\left[\cos(\omega t)[g_1'(y)B - g_2'(y)A + \sqrt{\frac{\omega}{2\nu_1}}(\sinh(e)\cos(e) + \sin(e)\cosh(e))(A^2 + B^2)] + \sin(\omega t)[-(g_1'(y)A + g_2'(y)B) + \sqrt{\frac{\omega}{2\nu_1}}(\sinh(e)\cos(e) - \sin(e)\cosh(e))(A^2 + B^2)]\right], \tag{3.67}$$

$$\tau_{1t}(y,t) = \sum_{m=1}^{\infty}\left(\frac{2\mu_1 U_0 \omega k_m^2}{\sqrt{v_1}(k_m^4+\omega^2)}\right)\left[\frac{F_2(k_m)\cos(k_m\frac{y}{\sqrt{v_1}}) + F_3(k_m)\sin(k_m\frac{y}{\sqrt{v_1}})}{F_4(k_m)}\right]\exp(-k_m^2 t), \quad (3.68)$$

respectively. Here, the constants $A$ and $B$ are as defined in (3.20) and (3.21). The functions $g_1(y)$ and $g_2(y)$ are as defined in (3.25) and (3.26). And $e$ is as defined in (3.30). Also, $F_2(k_m), F_3(k_m)$, and $F_4(k_m)$ are as defined in (3.33)–(3.35). Here primes denote differentiation with respect to $y$.

Again, the steady periodic and transient shear stresses in the upper fluid can be obtained by plugging expressions (3.63) and (3.64), respectively, in formula (2.28), yielding

$$\tau_{2s}(y,t) = \frac{\mu_2 U_0}{A^2+B^2}\left[(g_4'(y)A - g_3'(y)B)\cos(\omega t) + (g_3'(y)A + g_4'(y)B)\sin(\omega t)\right], \quad (3.69)$$

$$\tau_{2t}(y,t) = \sum_{m=1}^{\infty}\frac{2\mu_2 U_0 \omega k_m^2}{\sqrt{v_2}(k_m^4+\omega^2)}\left[\frac{\cos(k_m\frac{(H-y)}{\sqrt{v_2}})}{F_4(k_m)}\right]\exp(-k_m^2 t), \quad (3.70)$$

respectively. Here, the functions $g_3(y)$ and $g_4(y)$ are as defined in (3.27) and (3.28).

We can now determine steady-state and transient shear stresses at the oscillating plate by substituting $y = 0$ into Eqs. (3.67) and (3.68), respectively. And the results are

$$\tau_{1ws}(0,t) = \frac{\mu_1 U_0}{A^2+B^2}\sqrt{\frac{\omega}{2v_1}}[\cos(\omega t)[(-K_1+K_2)B - (K_1+K_2)A] - \sin(\omega t)[(-K_1+K_2)A + (K_1+K_2)B]], \quad (3.71)$$

$$\tau_{1wt}(0,t) = \sum_{m=1}^{\infty}\left(\frac{2\mu_1 U_0 \omega k_m^2}{\sqrt{v_1}(k_m^4+\omega^2)}\right)\left[\frac{F_2(k_m)}{F_4(k_m)}\right]\exp(-k_m^2 t), \quad (3.72)$$

respectively. Where $K_1$ and $K_2$ are as defined (3.44) and (3.45).

Again, We can evaluate steady-state and transient shear stresses at the stationary plate by putting $y = H$ in Eqs. (3.69) and (3.70), respectively, obtaining

$$\tau_{2ws}(H,t) = -\frac{\mu_2 U_0}{A^2+B^2}\sqrt{\frac{\omega}{2v_2}}[(A-B)\cos(\omega t) + (A+B)\sin(\omega t)], \quad (3.73)$$

$$\tau_{2wt}(H,t) = \sum_{m=1}^{\infty}\frac{2\mu_2 U_0 \omega k_m^2}{\sqrt{v_2}(k_m^4+\omega^2)}\left[\frac{\exp(-k_m^2 t)}{F_4(k_m)}\right]. \quad (3.74)$$

*3.2.2.3. Special case: single-layer limit*. The steady periodic and transient velocity fields for oscillatory Couette flow for a single-layer fluid, (3.48) and (3.49), have been deduced earlier in this section as a special case. The velocity fields correspond to the case where the plate oscillates as $U_0\cos(\omega t)$ (the cosine oscillations). We adopt the same procedure to deduce the steady-state and transient velocity fields for oscillatory Couette flow for a single-layer fluid related to the sine oscillations of the plate from the corresponding velocity fields for the lower fluid, (3.61) and (3.62). The steady periodic and transient velocity fields are

$$u_{ssc}(y,t) = \frac{U_0}{A_1^2+B_1^2}[(f_2(y)A_1 - f_1(y)B_1)\cos(\omega t) + (f_1(y)A_1 + f_2(y)B_1)\sin(\omega t)], \quad (3.75)$$

$$u_{tsc}(y,t) = -\sum_{m=1}^{\infty}\frac{2U_0\omega k_m\sqrt{v}}{k_m^4+\omega^2}\left[\frac{\sin(k_m\frac{(H-y)}{\sqrt{v}})}{H\cos(k_m\frac{(H)}{\sqrt{v}})}\right]\exp(-k_m^2 t), \quad (3.76)$$

respectively. Here, the functions $f_1(y)$ and $f_2(y)$ are as defined in Eqs. (3.50) and (3.51). The constants $A_1$ and $B_1$ are given by Eqs. (3.52) and (3.53). Also, $k_m$ can be found from Eq. (3.54).

It is worth mentioning here that the results deduced above for the single-layer fluid are consistent with those that can be obtained from Khaled and Vafai[4] for the same flow.

Note that the preceding steady-state and transient velocity fields for a single-layer fluid can also be deduced from the corresponding velocity fields for the upper fluid, (3.63) and (3.64), by adopting the procedure outlined earlier in this section.

We can now evaluate shear stresses at the oscillating and fixed plates related to the flow of the single-layer fluid. We can compute the shear stresses in the same manner as that we have adopted earlier in this section. The results for the steady periodic and transient shear stresses at the oscillating plate are

$$\tau_{cssc}(0,t) = \frac{\mu U_0}{A_1^2+B_1^2}[(B_2A_1 - A_2B_1)\cos(\omega t) + (A_2A_1 + B_2B_1)\sin(\omega t)], \quad (3.77)$$

$$\tau_{ctsc}(0,t) = \sum_{m=1}^{\infty}\frac{2\mu U_0\omega k_m^2}{H(k_m^4+\omega^2)}\exp(-k_m^2 t), \quad (3.78)$$

respectively. Here, $A_2$ and $B_2$ are as defined in Eqs. (3.57) and (3.58), respectively. The results for the steady-state and transient shear stresses at the stationary plate are

$$\tau_{cssc}(H,t) = -\frac{\mu U_0}{A_1^2+B_1^2}\sqrt{\frac{\omega}{2v}}[(A_1-B_1)\cos(\omega t) + (A_1+B_1)\sin(\omega t)], \quad (3.79)$$

$$\tau_{ctsc}(H,t) = \sum_{m=1}^{\infty}\frac{2\mu U_0\omega k_m^2}{k_m^4+\omega^2}\left[\frac{1}{H\cos(k_m\frac{H}{\sqrt{v}})}\right]\exp(-k_m^2 t), \quad (3.80)$$

respectively.

We note here that, as far as we are aware, complete wall shear stresses related to Couette flow due to the cosine or the sine oscillations of the plate have not been previously reported in the literature. A complete shear stress is the sum of steady periodic and transient shear stresses.

## 4. Results and illustrative examples

In this work, we have studied Stokes' second problem and oscillatory Couette flow for a two-layer fluid. In the Stokes' problem case, the fluid is bounded only by a oscillating plate that causes the fluid motion. In the Couette flow case, the fluid is confined between two parallel plates, one of which oscillates and induce the fluid motion. In both the cases, we have considered both the cosine and the sine oscillations of the plate. For both the Stokes' problem and the Couette flow, we have obtained analytical velocity fields consisting of transient and steady periodic parts for both the layers of fluids. The fluids have different viscosities, densities, and thicknesses. We have evaluated transient and steady-state shear stresses at the boundaries of the flows.

Consider the Stokes' second problem and the oscillatory Couette flow for the two-layer fluid where a layer of corn oil (lighter) lies over a layer of water (heavier). In both the Stokes' and Couette problems, the water rests on the oscillating plate. It should be noted that oil over water is encountered in many practical situations[30]. We can utilize the analytical results obtained in the previous sections for Stokes' second problem and oscillatory Couette flow for a two-layer fluid to get some physical insights into the particular flows we have considered here. For the Stokes' problem case, We show the effects of the forms and the frequency of oscillations on the time $t$ to reach a steady-state flow of each of the lower and the upper fluids. Also, for the case, we compare wall velocity with steady-state wall shear stress. Again, for both the Stokes' problem and the Couette flow cases, we show the effects of the forms of oscillations on the transient and steady-state wall shear stresses. Also, for both the Stokes and the Couette cases, we demonstrate oscillations in steady-state fluid velocities in both the lower and upper fluids.

For the particular problems considered here, the values of the parameters (in the cm-gram-second (cgs) system) are as follows[30]: the viscosity of the water $\mu_1 = 0.01$, the kinematic viscosity of the water $\nu_1 = 0.01$, the viscosity of the corn oil $\mu_2 = 0.2$, the kinematic viscosity of the corn oil $\nu_2 = 0.22$, the thickness of the water $h = 0.2$ , the distance between the plates (Couette flow) $H = 0.5$, and the plate velocity amplitude $U_0 = 2$. For the graphical representations of the results for the Couette flow, we have taken 20 terms of the infinite series representing the transient parts into account.

Note that henceforth, in this section, by the lower and upper fluids we mean the water and the corn oil, respectively.

### *4.1. Stokes' second problem for a two-layer fluid*

In Figs. 3 and 4, every panel depicts a starting velocity profile for a time and a steady-state velocity profile for the same time for a case of flow in the lower(water) or upper(corn oil) layer. We note here that a starting velocity field is the sum of steady-state and transient velocity fields. In each of the panels, the starting and steady-state velocity profiles are almost the same, implying that the transient has died out and the flow has attained steady-state. The figures show that the time required for a flow in the lower or upper layer to reach steady-state is much greater for the sine oscillations(the plate oscillates as $U_0 \sin(\omega t)$) than that for the cosine oscillations. It is also noticed from the figures that for any given value of $\omega$, the

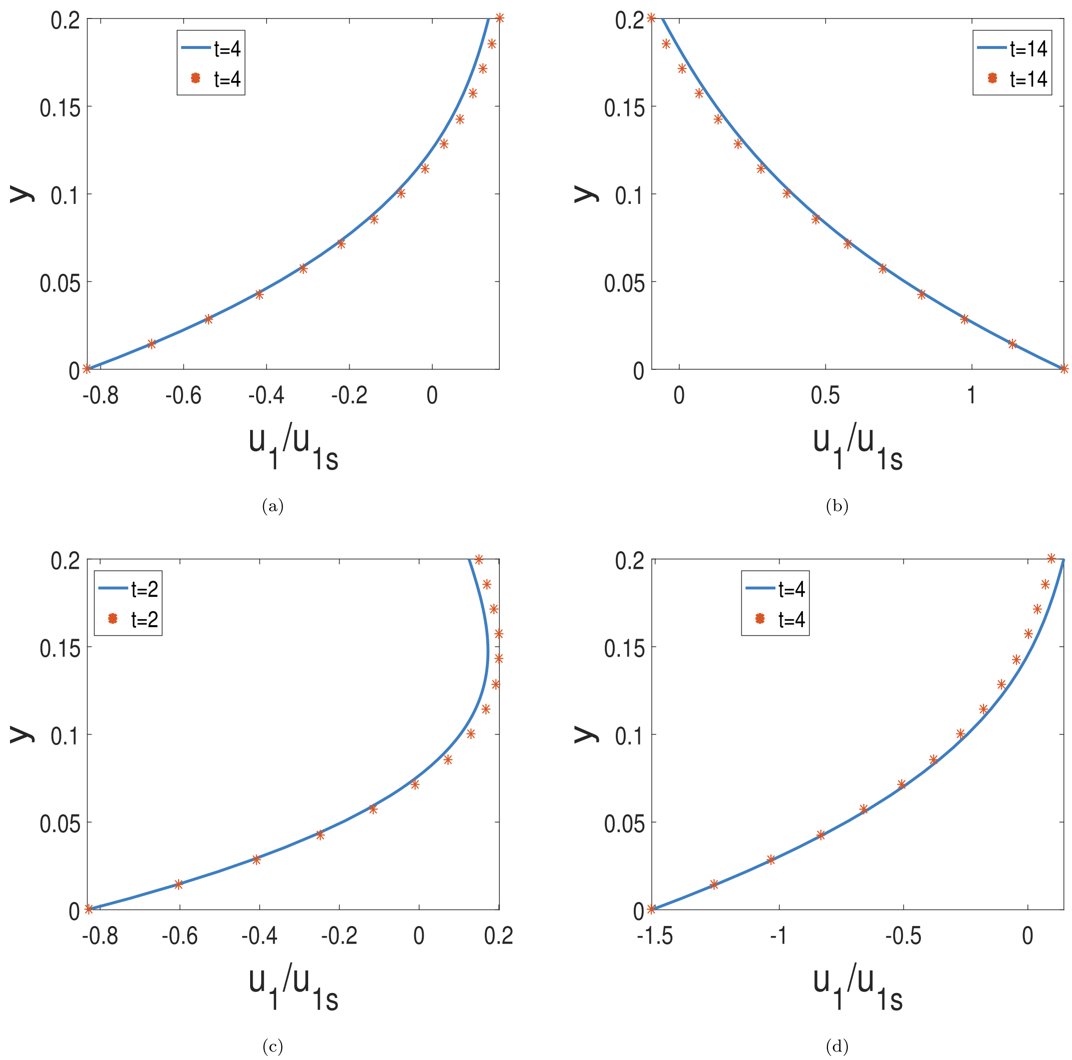



**Fig. 3** A profile for the starting velocity field (solid line) and a profile for the steady-state velocity field (line of asterisks) for the lower fluid when $U_0 = 2, h = 0.2, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$: (a) the plate oscillates as $U_0 \cos(\omega t)$, with $\omega = 0.5$, (b) the plate oscillates as $U_0 \sin(\omega t)$, with $\omega = 0.5$, (c) the plate oscillates as $U_0 \cos(\omega t)$, with $\omega = 1$, and (d) the plate oscillates as $U_0 \sin(\omega t)$, with $\omega = 1$. (Stokes' problem).

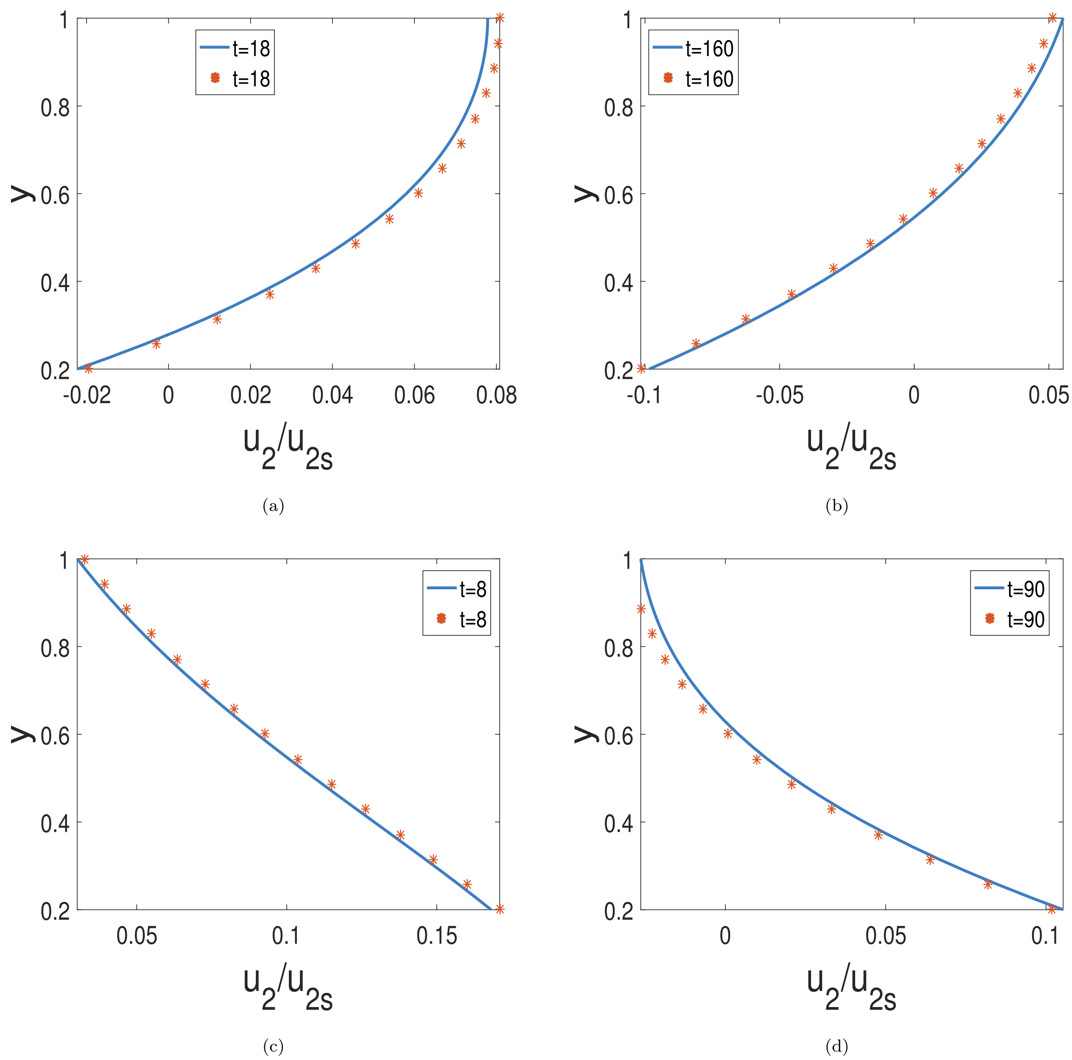
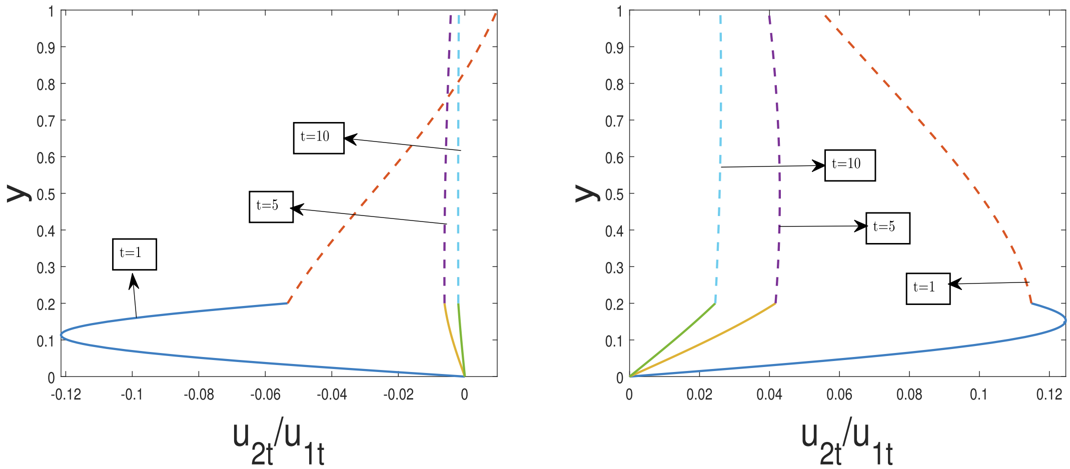



**Fig. 4** A profile for the starting velocity field (solid line) and a profile for the steady-state velocity field (line of asterisks) for the upper fluid when $U_0 = 2, h = 0.2, \mu_1 = 0.01, v_1 = 0.01, \mu_2 = 0.2, v_2 = 0.22$: (a) the plate oscillates as $U_0 \cos(\omega t)$, with $\omega = 0.5$, (b) the plate oscillates as $U_0 \sin(\omega t)$, with $\omega = 0.5$, (c) the plate oscillates as $U_0 \cos(\omega t)$, with $\omega = 1$, and (d) the plate oscillates as $U_0 \sin(\omega t)$, with $\omega = 1$. (Stokes' problem).

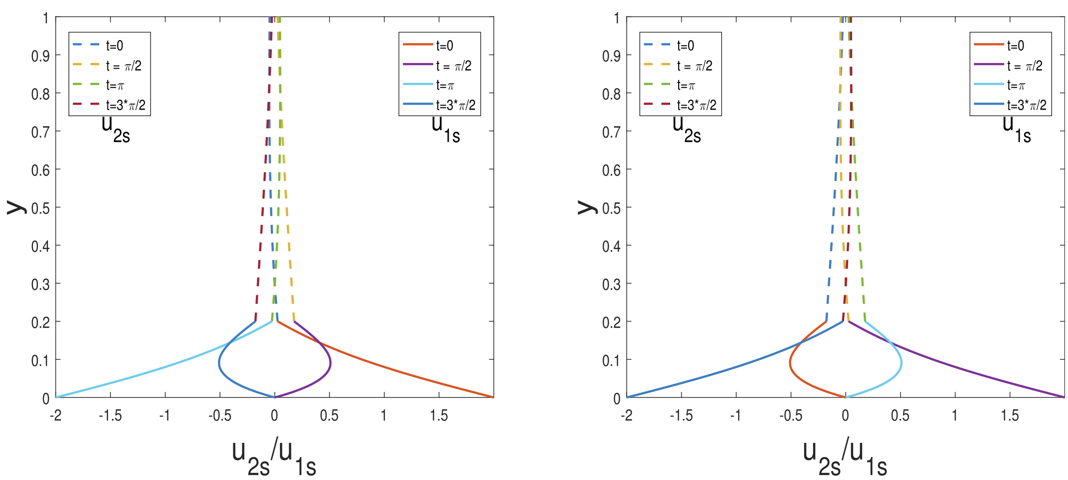
oscillations frequency of the plate, the time needed to reach steady-state velocity in the lower fluid is much less than that in the upper fluid. The finding holds true for both forms of oscillations of the plate. As seen from Fig. 3, when $\omega = 0.5$, the fluid motion in the lower layer becomes steady periodic around $t = 4$ when the plated is subjected to the cosine oscillations, and for the sine oscillations, the required time is $t = 14$. Again, when $\omega = 1$, the fluid flow in the lower layer attains steady-state around $t = 2$ and $t = 4$ for the cosine and the sine oscillations of the plate, respectively, as noticed from the same figure. Regarding the upper layer of fluid, Fig. 4 shows that when $\omega = 0.5$, the flows corresponding to the cosine and the sine oscillations of the plate attain steady-state about $t = 18$ and $t = 160$, respectively. Again, when $\omega = 1$, the upper fluid flow reaches steady-state around $t = 8$ when the plate is subjected to the cosine oscillations, and for the sine oscillations of the plate, the required time is $t = 90$, as noticed from the same figure.

Fig. 5 gives transient velocity profiles for three given times for the lower and upper fluids. Fig. 5a and b illustrate profiles corresponding to the cosine and the sine oscillations of the plate, respectively. It is noticed from the figure that the transient velocity (absolute value) decreases rapidly at initial stages but after some time the rate of decreasing with respect to time slows down. It is observed for both forms of oscillations of the plate. Also, the following pieces of information are obtained from the figure. When the plate oscillates as $U_0 \cos(\omega t)$(Fig. 5a), at $t = 1$, the maximum transient velocity (absolute value) in the lower fluid is slightly greater than 0.12 and it occurs at just over $y = 0.1$. At the same time, the maximum transient velocity (absolute value) in the upper fluid is slightly greater than 0.05 and it occurs at the interface of the fluids($y = 0.2$). Again, when the plate oscillates as $U_0 \sin(\omega t)$(Fig. 5b), at $t = 1$, the maximum transient velocity (absolute value) in the lower fluid is 0.125, it occurs at $y = 0.15$. At the same time, the maximum transient velocity




(a) (b)

**Fig. 5** Profiles for the transient velocity fields for the lower (solid lines) and upper (broken lines) fluids when $U_0 = 2, h = 0.2, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$: (a) the plate oscillates as $U_0 \cos(\omega t)$, and (b) the plate oscillates as $U_0 \sin(\omega t)$. (Stokes' problem).




(a) (b)

**Fig. 6** Steady-state velocity profiles for the lower (solid lines) and upper (broken lines) fluids when $U_0 = 2, h = 0.2, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$: (a) the plate oscillates as $U_0 \cos(\omega t)$, and (b) the plate oscillates as $U_0 \sin(\omega t)$. (Stokes' problem) (Use color in print).

(absolute value) in the upper fluid is slightly greater than 0.11 and it occurs at the interface of the fluids.

Fig. 6 shows steady periodic velocity profiles in the lower and upper fluids. Fig. 6a and b depict profiles corresponding to the cosine and the sine oscillations of the plate, respectively. Oscillations in the fluid velocities in both the layers are noticed from the figure, as expected.

Fig. 7 illustrates transient wall shear stresses related to the cosine and the sine oscillations of the plate. The figure shows that at very small times the magnitude of transient wall shear stress for the cosine oscillations of plate is significantly bigger than that corresponding to the sine oscillations. However, in both the cases the transient wall shear stress dies out at around $t = 1.5$.

Fig. 8 depicts steady-state wall shear stresses related to the cosine and the sine oscillations of the plate. Two intervals of time have been considered: a) the duration of motion $t \in [0, 30]$, and b) the duration of motion $t \in [0, 100]$. It is noticed from the figure that for all times, excepting for very small times, the steady-state wall shear stresses corresponding to the cosine and the sine oscillations of the plate have similar amplitudes with a phase difference.

Fig. 9 compares steady-state wall shear stress with wall velocity. Fig. 9a considers the cosine oscillations of the plate whereas Fig. 9b do the sine oscillations of the plate. It is seen from the figure that for both the cosine and the sine oscillations of the plate, wall shear stress lags behind wall velocity. This can also be seen from expressions for steady-state wall shear stresses, which can be obtained from (2.29) and (2.34), corresponding to wall velocity $U_0 \cos(\omega t)$ and $U_0 \sin(\omega t)$, respectively.

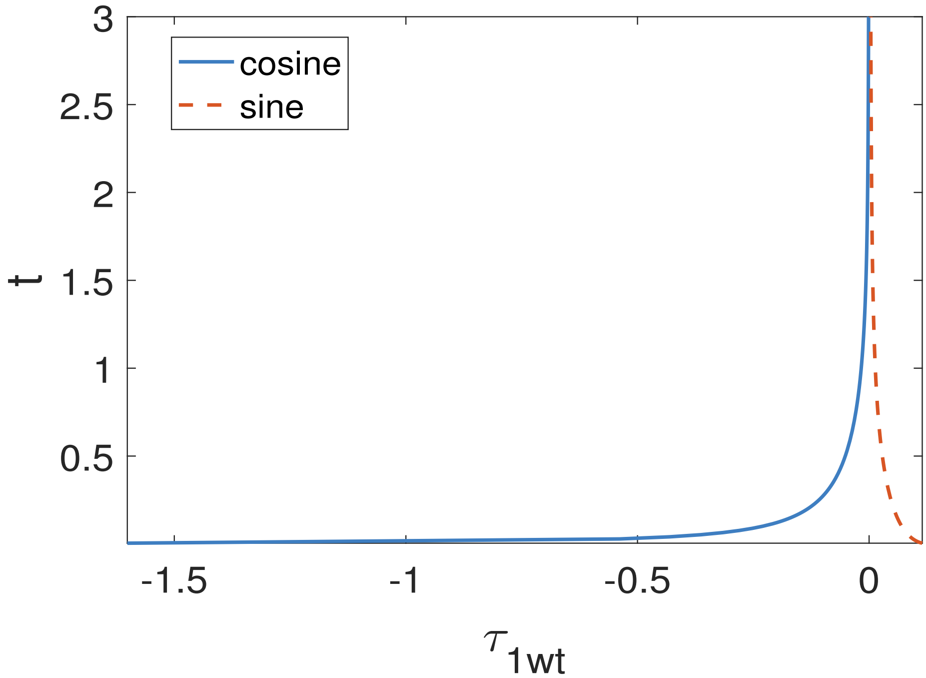



**Fig. 7** The transient shear stress at the plate when $U_0 = 2, h = 0.2, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$. (Stokes' problem).

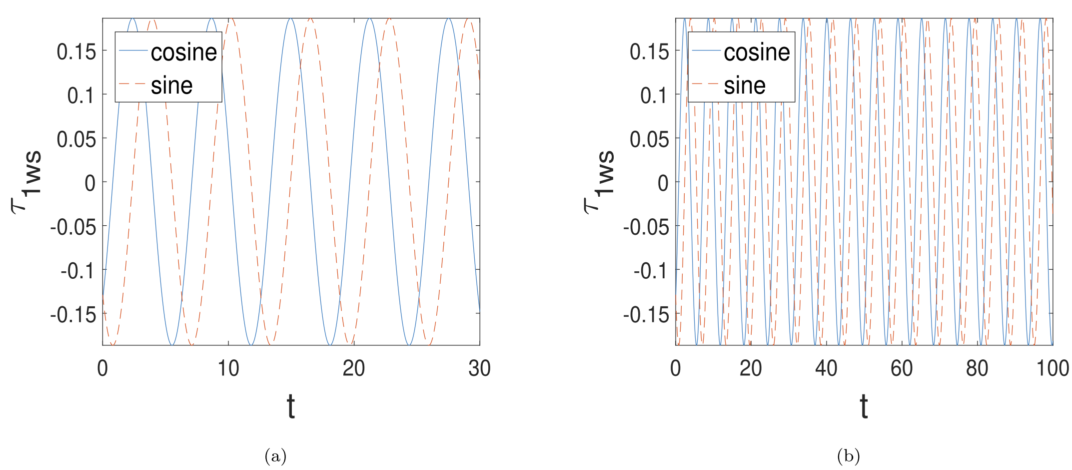



**Fig. 8** The steady-state shear stress at the plate when $U_0 = 2, h = 0.2, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$: (a) the duration of motion $t \in [0, 30]$, and (b) the duration of motion $t \in [0, 100]$. (Stokes' problem).

### *4.2. Oscillatory Couette flow for a two-layer fluid*

Fig. 10 illustrates transient velocity profiles for the lower(water) and upper(corn oil) fluids for the cosine and the sine oscillations of the plate. Profiles corresponding to the cosine oscillations of the plate are presented in Fig. 10a, and those related to the sine oscillations are depicted in Fig. 10b. It is noticed from the figure that the transient velocities for both the lower and upper fluids die out very rapidly for both the forms of oscillations of the plate. The transient velocities disappear rapidly because of exponentials in their expressions, (3.31),(3.32), (3.62), and (3.64). On Fig. 10a, at $t = 0.1$, the maximum velocity (absolute value) in the lower fluid is about 0.3, and it occurs around $y = 0.05$ and $y = 0.14$. At the same time, the maximum velocity (absolute value) in the upper fluid is approximately 0.1, which occurs at $y = 0.2$ (i.e. at the interface of the fluids). On Fig. 10b, at $t = 0.1$, the maximum velocity (absolute value) in the lower fluid is approximately 0.014, and it occurs about $y = 0.05$ and $y = 0.14$. At the same time, the maximum velocity (absolute value) in the upper fluid is 0.005, which occurs at the interface of the fluids.

Fig. 11 shows steady periodic velocity profiles for the lower and upper fluids. Fig. 11a and b illustrate profiles related to the cosine and the sine oscillations of the plate, respectively. Oscillations in fluid velocities for both the fluids are noticed from the figure, as expected.

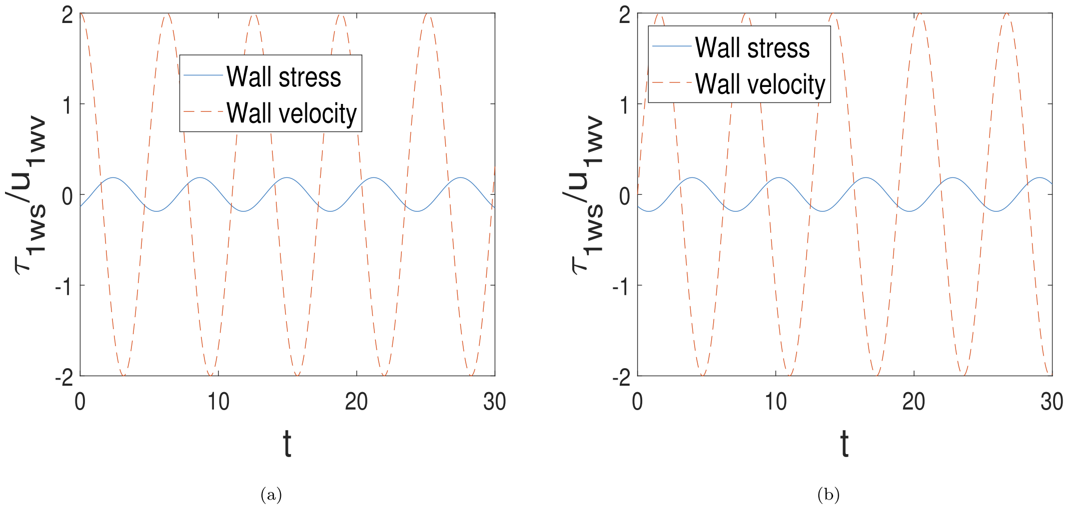



**Fig. 9** Wall velocity (broken line) and steady-state wall shear stress (solid line) when $U_0 = 2, h = 0.2, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$: (a) the plate oscillates as $U_0 \cos(\omega t)$, and (b) the plate oscillates as $U_0 \sin(\omega t)$. (Stokes' problem).

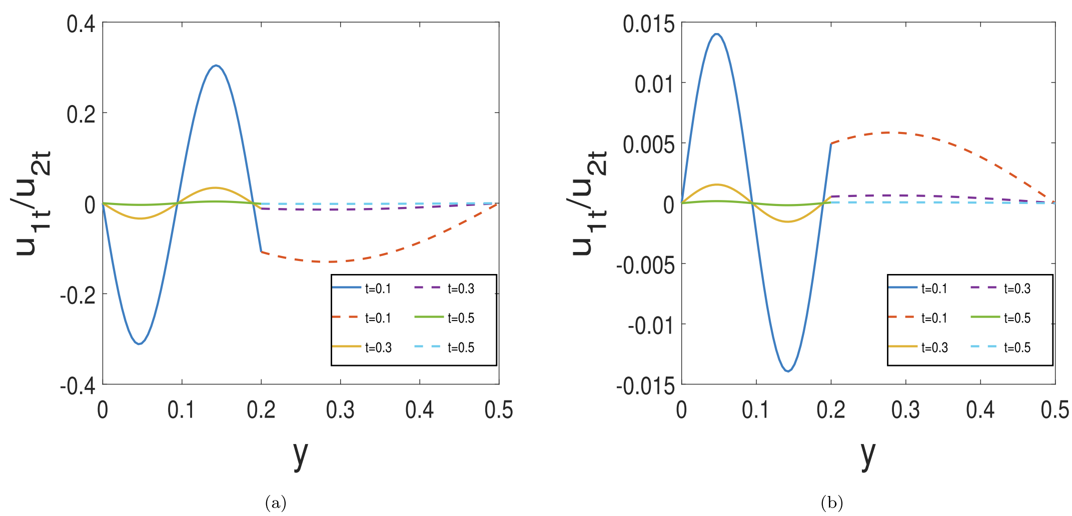


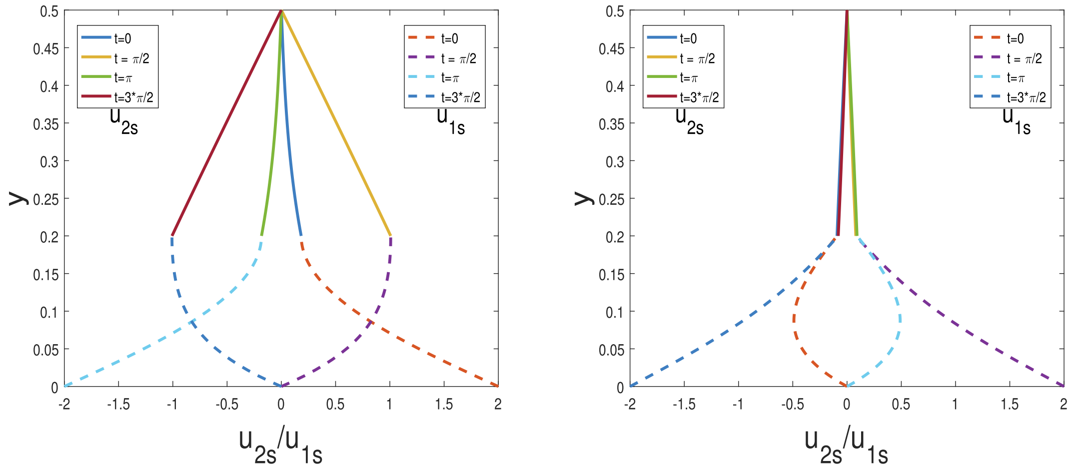

**Fig. 10** Profiles for the transient velocity fields for the lower (solid lines) and upper (broken lines) fluids when $U_0 = 2, h = 0.2, H = 0.5, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$: (a) the plate oscillates as $U_0 \cos(\omega t)$, and (b) the plate oscillates as $U_0 \sin(\omega t)$. (Couette flow) (Use color in print).

Fig. 12 illustrates transient shear stresses at the oscillating and fixed plates. Both the forms of oscillations of the plate are considered in the figure. While panel (a) illustrates transient shear stress at the oscillating plate, panel (b) depicts transient shear stress at the fixed plate. It is noticed from the figure that at both the plates, the transient shear stress corresponding to the sine oscillations of the plate is zero for all values of time $t$. Contrary to the sine oscillations, the magnitudes of the transient shear stresses at the plates related to the cosine oscillations of the plate are quite significant for very small values of time $t$. However, these transient shear stresses related to the cosine oscillations of the plate disappear very rapidly.

Figs. 13 and 14 illustrate steady-state shear stresses at the oscillating and stationary plates, respectively. Both the cosine and the sine oscillations of the plate are considered in the figures. In both the figures, two intervals of time $t$ have been considered: $t \in [0, 30]$(panel (a)) and $t \in [0, 100]$ (panel (b)). The steady periodic shear stresses at the oscillating plate for the cosine and the sine oscillations of the plate have similar amplitudes with a phase difference for all the times, with the exceptions for very small times, as seen from Fig. 13. The same pattern is observed for steady-state shear stresses at the fixed plate for the cosine and the sine oscillations of the plate, as noticed from Fig. 14.

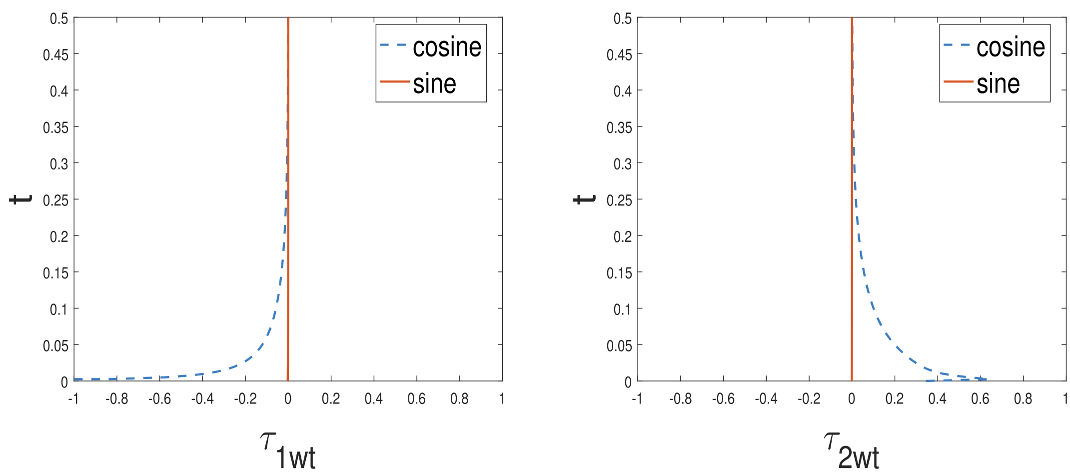
## 5. Conclusions

In this paper, we have mathematically analyzed the unsteady motion of a two-layer fluid, in both semi-infinite (Stokes' problem) and wall-bounded (Couette flow) domains, caused by an oscillating flat plate. Both cosine and sine oscillations of the plate have been considered. Initially, the fluids and the plate have been at rest and then suddenly, the plate starts to oscillate




(a) (b)

**Fig. 11** Steady-state velocity profiles for the lower (broken lines) and upper (solid lines) fluids when $U_0 = 2, h = 0.2, H = 0.5, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$: (a) the plate oscillates as $U_0 \cos(\omega t)$, and (b) the plate oscillates as $U_0 \sin(\omega t)$. (Couette flow) (Use color in print).

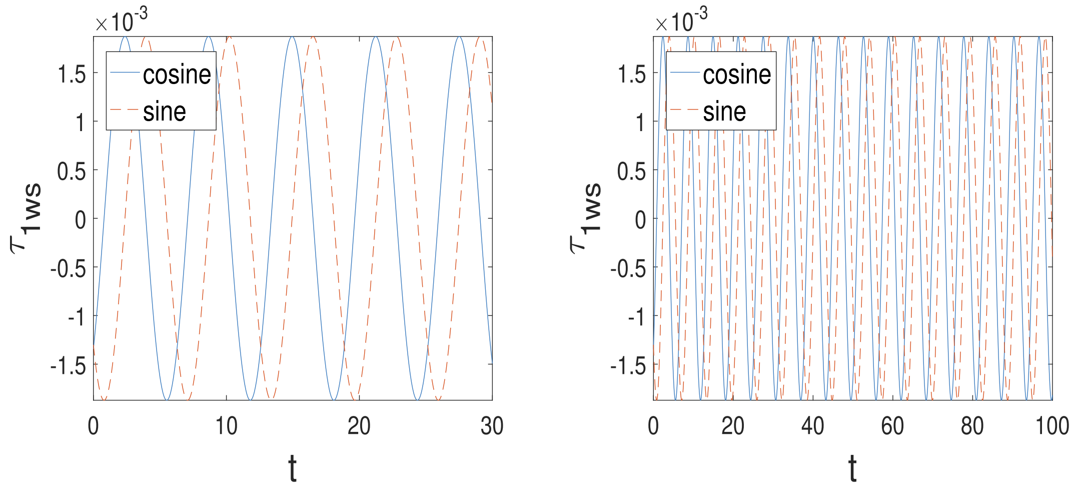



(a) (b)

**Fig. 12** The transient shear stresses at the oscillating and fixed plates when $U_0 = 2, h = 0.2, H = 0.5, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$: (a) oscillating plate, and (b) stationary plate. (Couette flow).

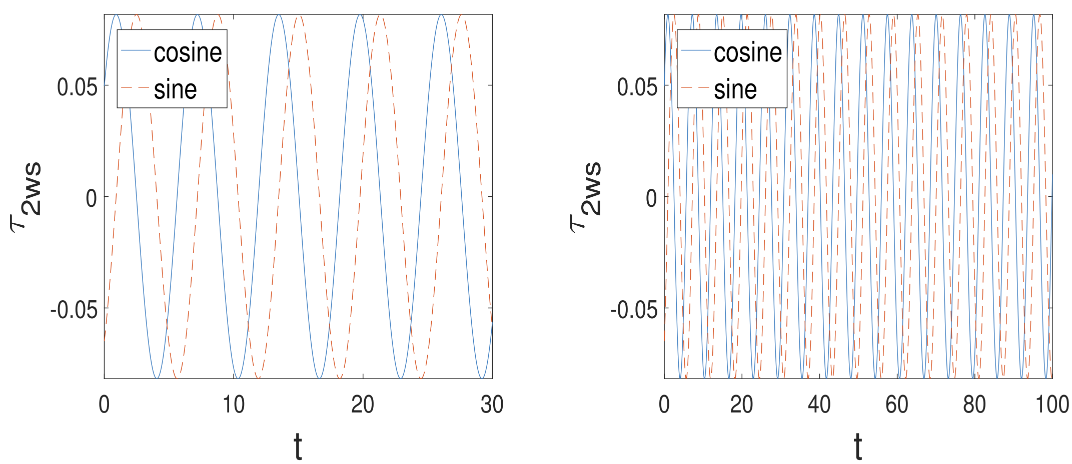
in its own plane. The Laplace transform method has been employed to solve the associated initial-boundary value problems. And the Bromwich inversion integral and Cauchy's residue theorem have been utilized to find inverse Laplace transforms of the velocity fields. For both the layers of fluid, analytical expressions for starting and steady-state velocity fields have been obtained. A starting velocity field is the sum of the transient and steady-state velocity fields and valid for small values of time $t$. Explicit expressions for transient and steady-state velocity fields have been presented. The results for the velocity fields presented in this paper are new and complete. And the results retrieve related previously reported results for single-layer fluid flows (can be found in [4]) as special cases. Also, we have computed transient and steady-state shear stresses at the boundaries of the flows. The results for shear stresses will have been reported in the literature for the first time.

The mathematical results have been illustrated taking water and corn oil as lower and upper fluids, respectively. The illustrations have helped us gain some physical insights into the flows of the particular problems considered. For the Stokes' problem case, it has been found that in both the layers of fluid, the time required to reach a steady-state flow when the plate is subjected to cosine oscillations is much less than that when the plate is subjected to sine oscillations. Again, for a given oscillation frequency of the plate, the lower fluid which is adjacent to the plate attains a steady-state flow much earlier than the upper fluid. It holds true for both the cosine and the sine oscillations of the plate. Again, irrespective of the form of oscillations of the plate and true for both the lower and upper fluids, when the frequency of oscillations increases, the time to reach a steady-state flow decreases. Again, the steady-state shear stresses at the plate related to cosine and sine oscillations of the plate have similar amplitudes with a phase difference for




(a) (b)

**Fig. 13** The steady-state shear stress at the oscillating plate when $U_0 = 2, h = 0.2, H = 0.5, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$: (a) the duration of motion $t \in [0, 30]$, and (b) the duration of motion $t \in [0, 100]$. (Couette flow).




(a) (b)

**Fig. 14** The steady-state shear stress at the fixed plate when $U_0 = 2, h = 0.2, H = 0.5, \mu_1 = 0.01, \nu_1 = 0.01, \mu_2 = 0.2, \nu_2 = 0.22$, and $\omega = 1$: (a) the duration of motion $t \in [0, 30]$, and (b) the duration of motion $t \in [0, 100]$. (Couette flow).

all times, excepting very small times. For the Couette flow case, it is found that for both cosine and sine oscillations of the plate, the transient velocity disappears very rapidly in both the layers of fluid. Again, at both the oscillating and stationary plates, steady-state shear stresses have similar amplitudes with a phase difference for all times, excepting very small times. Note that in the particular problem, we have considered the case where the thickness of the lower fluid is less than that of the upper fluid.

It is our believe that the present study deepen our understanding of the motion of a two-layer viscous fluid caused by an oscillating wall in an engineering application. However, the results of the study may not apply to a case where deviation of the interface (between the layers of fluid) from flat shape is substantial. But it is worth noting that the analytical results obtained in this paper could be used for validation of future numerical works dealing with the motion of a two-layer fluid induced by an oscillating wall with wavy interface between the layers of fluid. Moreover, the present study may provide a basis for some important future researches as mentioned in the introduction. Furthermore, the present work may also be applicable to heat conduction in a two-layer composite solid subject to sinusoidal temperature variation on the surface.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Acknowledgements

The authors would like to thank the referees for their valuable comments and suggestions which have led to a number of improvements.

**Appendix A. Evaluation of inverse Laplace transform using Bromwich inversion integral**

Here we evaluate $\mathscr{L}^{-1}\left(\frac{s\exp(-a\sqrt{s})}{s^2+\omega^2}\right), a > 0$, where $\mathscr{L}^{-1}$ is inverse Laplace transform operator. The inverse transform has been utilized to obtain the velocity fields for the lower and upper fluids, (2.25) and (2.26), in Section 2.2.1.

The Laplace transform of a function $f(t)$ is defined as follows:

$$F(s) = \mathscr{L}(f(t)) = \int_0^\infty f(t)\exp(-st)dt, \tag{A.1}$$

where $\mathscr{L}$ is Laplace transform operator, and $s$ is transform variable.

For time $t > 0$, the inverse Laplace transform of $F(s)$ is given by the following formula:

$$f(t) = \mathscr{L}^{-1}(F(s)) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)\exp(st)ds. \tag{A.2}$$

The inversion formula is called the Bromwich inversion integral[51,52]. In the formula, $\gamma$ is a real number, and it must be chosen such that all the singularities of $F(s)$ (poles, branch points or essential singularities) lie to the left of the line $s = \gamma$ in the complex s-plane. The integration in the formula is to be evaluated along the line $s = \gamma$. However, in practice, the integration is performed along a closed contour composed of the line $s = \gamma$ and a circular arc on the left of the line. This is done in order to facilitate the use of Cauchy's residue theorem[54].

For the case in hand,

$$F(s) = \left(\frac{s\exp(-a\sqrt{s})}{s^2+\omega^2}\right). \tag{A.3}$$

$F(s)$ has simple poles at $s = -i\omega$ and $s = i\omega$. Also, since $F(s)$ contains a fractional power, $s^{\frac{1}{2}}$, the point $s = 0$ is a branch point. We now consider the contour integral

$$\frac{1}{2\pi i}\oint_C F(s)\exp(st)ds, \tag{A.4}$$

where $C$ is the contour of Fig. (15). The contour $C$ is a keyhole contour. We have drawn the keyhole contour in order to exclude the branch cut along the negative real axis. The contour $C$ is composed of line $AB$, circular arc $BE$, line $EF$, a small circle around the origin $O$ of radius $\epsilon$, line $GH$, and circular arc $HA$. Arcs $BE$ and $HA$ are arcs of a circle of radius $R$ with center at the origin $O$. Now, it follows from formula (A.2) that

$$f(t) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma-iL}^{\gamma+iL} F(s)\exp(st)ds, \tag{A.5}$$

since $L = \sqrt{R^2-\gamma^2}$. Again, it follows from (A.5) that

$$f(t) = \lim_{\substack{R\to\infty\\ \epsilon\to 0}}\left[\frac{1}{2\pi i}\oint_C F(s)\exp(st)ds - \frac{1}{2\pi i}\left\{\int_{BE} + \int_{EF} + \int_{FG} + \int_{GH} + \int_{HA}\right\}F(s)\exp(st)ds\right]. \tag{A.6}$$

Now, on the arcs $BE$ and $HA, F(s) \to 0$ exponentially as $R \to \infty$, since real part of $s^{\frac{1}{2}}$ is positive. Therefore,

$$\int_{BE} F(s)\exp(st)ds = 0 \quad \text{and} \quad \int_{HA} F(s)\exp(st)ds = 0. \tag{A.7}$$

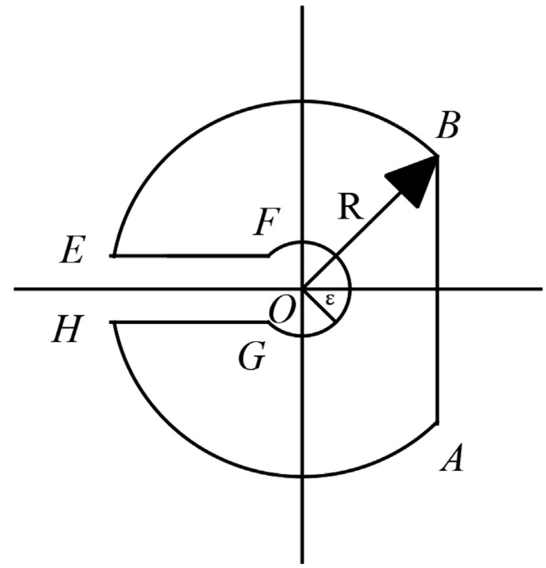



**Fig. 15** Bromwich contour integral.

We now compute

$$\int_{FG} F(s)\exp(st)ds.$$

We use $s = \epsilon e^{i\theta}$, where $\theta$ runs from $\pi$ to $-\pi$, to parameterize the small circle $FG$. On this circle, $s^{\frac{1}{2}} = \epsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}}$, so that

$$\int_{FG}\frac{s\exp(st-as^{\frac{1}{2}})}{s^2+\omega^2}ds = \int_\pi^{-\pi}\frac{i\epsilon^2e^{2i\theta}\exp(\epsilon e^{i\theta}t - a\epsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}})}{\epsilon^2e^{2i\theta}+\omega^2}d\theta.$$

Therefore, as $\epsilon \to 0$,

$$\int_{FG}\frac{s\exp(st-as^{\frac{1}{2}})}{s^2+\omega^2}ds = 0. \tag{A.8}$$

To evaluate the integrals along the lines $EF$ and $GH$, we parameterize the lines using $s = r\exp(i\pi) = -r$ and $s = r\exp(-i\pi) = -r$, respectively. Along the line $EF, s^{\frac{1}{2}} = r^{\frac{1}{2}}\exp(\frac{i\pi}{2}) = ir^{\frac{1}{2}}$, and along the line $GH, s^{\frac{1}{2}} = r^{\frac{1}{2}}\exp(\frac{-i\pi}{2}) = -ir^{\frac{1}{2}}$. Therefore, as $\epsilon \to 0$ and $R \to \infty$,

$$\int_{EF}\frac{s\exp(st-as^{\frac{1}{2}})}{s^2+\omega^2}ds = -\int_0^\infty\frac{r\exp(-rt-iar^{\frac{1}{2}})}{r^2+\omega^2}dr$$

and

$$\int_{GH}\frac{s\exp(st-as^{\frac{1}{2}})}{s^2+\omega^2}ds = \int_0^\infty\frac{r\exp(-rt+iar^{\frac{1}{2}})}{r^2+\omega^2}dr.$$

Therefore, we have

$$\left\{\int_{EF} + \int_{GH}\right\}\frac{s\exp(st-as^{\frac{1}{2}})}{s^2+\omega^2}ds = 2i\int_0^\infty\frac{r\exp(-rt)\sin(a\sqrt{r})}{r^2+\omega^2}dr. \tag{A.9}$$

Using (A.7), (A.8), and (A.9) in (A.6), we have

$$f(t) = \lim_{\substack{R\to\infty\\ \epsilon\to 0}}\left[\frac{1}{2\pi i}\oint_C F(s)\exp(st)ds\right] - \frac{1}{\pi}\int_0^\infty \times\frac{r\exp(-rt)\sin(a\sqrt{r})}{r^2+\omega^2}dr. \tag{A.10}$$

Now, as $R \to \infty$, all the poles of the integrand of the integration around the closed contour $C$ lie within $C$. The integrand has simple poles at $s = -i\omega$ and $s = i\omega$, noting that $F(s)$ is given by Eq. (A.3). Therefore, according to Cauchy's residue theorem[54],

$$\begin{aligned}\oint_C F(s)\exp(st)ds &= \oint_C \frac{s\exp(st-as^{\frac{1}{2}})}{s^2+\omega^2}ds \\ &= 2\pi i\left[\lim_{s\to -i\omega}\left\{(s+i\omega)\frac{s\exp(st-as^{\frac{1}{2}})}{s^2+\omega^2}\right\}\right. \\ &\left. +\lim_{s\to i\omega}\left\{(s-i\omega)\frac{s\exp(st-as^{\frac{1}{2}})}{s^2+\omega^2}\right\}\right] \\ &= 2\pi i \exp\left(-a\sqrt{\tfrac{\omega}{2}}\right)\cos\left(\omega t - a\sqrt{\tfrac{\omega}{2}}\right).\end{aligned} \quad (A.11)$$

We now use (A.11) in (A.10) to obtain the following result:

$$\mathscr{L}^{-1}\left(\frac{s\exp(-a\sqrt{s})}{s^2+\omega^2}\right) = \exp\left(-a\sqrt{\frac{\omega}{2}}\right)\cos\left(\omega t - a\sqrt{\frac{\omega}{2}}\right) - \frac{1}{\pi}\int_0^\infty \frac{r\exp(-rt)\sin(a\sqrt{r})}{r^2+\omega^2}dr, \quad (A.12)$$

for $a > 0$.

## Appendix B. Deduction of velocity field for oscillatory Couette flow for a single-layer fluid as a special case, when the plate oscillates as $U_0\cos(\omega t)$

### B.1. Deduction from the velocity field for the lower fluid

Here we provide hints about the deductions of the steady periodic and transient velocity fields for oscillatory Couette flow for a single-layer fluid, (3.48) and (3.49), respectively, in Section 3.2.1. The deductions are made from the steady periodic and transient velocity fields for the lower fluid, (3.18) and (3.31), in the same section. The steady periodic velocity field for the lower fluid, (3.18), contains $A, B, g_1(y)$, and $g_2(y)$, which are defined in Eqs. (3.20),(3.21),(3.25), and (3.26), respectively. If we let $h = H, \mu_1 = \mu_2 = \mu$(say the viscosity of the single-layer fluid), and $\nu_1 = \nu_2 = \nu$(say the kinematic viscosity of the single-layer fluid) in Eqs. (3.20),(3.21),(3.25), and (3.26), we have

$$A = \cos b_1 \sinh b_1, \quad (B.1)$$

$$B = \sin b_1 \cosh b_1, \quad (B.2)$$

$$g_1(y) = -\sin b_2 \cosh b_2 \sin b_1 \sinh b_1 + \cos b_2 \sinh b_2 \cos b_1 \cosh b_1, \quad (B.3)$$

$$g_2(y) = \sin b_2 \cosh b_2 \cos b_1 \cosh b_1 + \cos b_2 \sinh b_2 \sin b_1 \sinh b_1, \quad (B.4)$$

$$\text{where} \quad b_1 = \sqrt{\frac{\omega}{2\nu}}H, \quad \text{and} \quad b_2 = \sqrt{\frac{\omega}{2\nu}}y. \quad (B.5)$$

If we substitute (B.1)–(B.4) into Eq. (3.18), we obtain the steady periodic velocity field for oscillatory Couette flow for the single-layer fluid, (3.48).

Again, the transient velocity field for the lower fluid, (3.31), contains $F_2(k_m), F_3(k_m)$, and $F_4(k_m)$, which are defined in Eqs. (3.33)–(3.35), respectively. If we put $h = H, \mu_1 = \mu_2 = \mu$(say the viscosity of the single-layer fluid), and $\nu_1 = \nu_2 = \nu$(say the kinematic viscosity of the single-layer fluid) in Eqs. (3.33)–(3.35), we obtain

$$F_2(k_m) = \cos(k_m \frac{H}{\sqrt{\nu}}), \quad (B.6)$$

$$F_3(k_m) = \sin(k_m \frac{H}{\sqrt{\nu}}), \quad (B.7)$$

$$F_4(k_m) = \frac{H}{\sqrt{\nu}}\cos(k_m \frac{H}{\sqrt{\nu}}). \quad (B.8)$$

Note that here, $k_m$ is as defined in (3.54). Substituting Eqs. (B.6)–(B.8) into Eq. (3.31), we obtain the transient velocity field for oscillatory Couette flow for the single-layer fluid, (3.49).

### B.2. Deduction from the velocity field for the upper fluid

Here we give hints on the deductions of the steady periodic and transient velocity fields for oscillatory Couette flow for a single-layer fluid, Eqs. (3.48) and (3.49) in section (3.2.1). The deductions are made from the corresponding results for the upper fluid, Eqs. (3.19) and (3.32) in the same section. The steady-state velocity field for the upper fluid, (3.19), contains $A, B, g_3(y)$, and $g_4(y)$, which are defined in Eqs. (3.20), (3.21), (3.27), and (3.28), respectively. If we let $fh = 0$(meaning that the lower fluid ceases to exist), $\mu_1 = \mu_2 = \mu$(say the viscosity of the single-layer fluid), and $\nu_1 = \nu_2 = \nu$(say the kinematic viscosity of the single-layer fluid) in Eqs. (3.20),(3.21),(3.27), and (3.28), $A$ and $B$ reduce to those defined in Eqs. (B.1) and (B.2), and $g_3(y)$ and $g_4(y)$ reduce to as follows:

$$g_3(y) = \cos(b_1 - b_2)\sinh(b_1 - b_2), \quad (B.9)$$

$$g_4(y) = \cosh(b_1 - b_2)\sin(b_1 - b_2), \quad (B.10)$$

where $b_1$ and $b_2$ are as defined in (B.5).

Substitutions of Eqs. (B.1), (B.2), (B.9), and (B.10) into Eq. (3.19) results in the steady periodic velocity field for oscillatory Couette flow for the single-layer fluid, (3.48).

Again, the transient velocity field for the upper fluid, (3.32), contains $F_4(k_m)$, which is defined in (3.35). If we put $h = 0$(meaning that the lower fluid ceases to exist), $\mu_1 = \mu_2 = \mu$(say the viscosity of the single-layer fluid), and $\nu_1 = \nu_2 = \nu$(say the kinematic viscosity of the single-layer fluid) in Eq. (3.35), $F_4(k_m)$ reduces to that defined in Eq. (B.8). Substitution of Eq. (B.8) into (3.32) yields the transient velocity field for oscillatory Couette flow for the single-layer fluid, (3.49).

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